電気回路演習 II 第 12 回 (平成 18 年 1 月 26 日 (金))

演習

1. (1),(2),(3) のラプラス変換を求めなさい

(1) $f_1(t) = \begin{cases} 0 & (0 < t < a) \\ 1 & (a \leq < t < b) \\ 0 & (b \leq t) \end{cases}$

(2) $f_2(t) = \cosh \beta t$

(3) $f_3(t) = \cos(\omega t + \theta)$

2. (4),(5) のラプラス逆変換を求めなさい

(4) $F_4(s) = \dfrac{6}{(s+2)^2 + 9}$

(5) $F_5(s) = \dfrac{1}{s^3 + 3s^2}$

解答

(1)

$$
\begin{aligned}
F_1(s) &= \int_0^\infty f_1(t) e^{-st} dt = \int_0^a 0 \cdot e^{-st} dt + \int_a^b e^{-st} dt + \int_b^\infty 0 \cdot e^{-st} dt = \left[-\frac{e^{-st}}{s} \right]_a^b \\
&= \frac{e^{-as} - e^{-bs}}{s}
\end{aligned}
$$

(2)

$$
\begin{aligned}
F_2(s) &= \int_0^\infty \cosh \beta t \cdot e^{-st} dt = \int_0^\infty \frac{e^{-(s-\beta)t} + e^{-(s+\beta)t}}{2} dt = \frac{1}{2} \left[-\frac{e^{-(s-\beta)t}}{s-\beta} - \frac{e^{-(s+\beta)t}}{s+\beta} \right]_0^\infty \\
&= \frac{1}{2} \left(\frac{1}{s-\beta} + \frac{1}{s+\beta} \right) = \frac{s}{s^2 - \beta^2}
\end{aligned}
$$

(3)

$$
\begin{aligned}
F_3(s) &= \int_0^\infty \cos(\omega t + \theta) \cdot e^{-st} dt = \frac{1}{2} \int_0^\infty \left\{ e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)} \right\} e^{-st} dt \\
&= \frac{1}{2} \int_0^\infty \left\{ e^{j\theta} \cdot e^{-(s - j\omega)t} + e^{-j\theta} \cdot e^{-(s + j\omega)t} \right\} dt \\
&= \frac{1}{2} \left[-\frac{e^{j\theta} \cdot e^{-(s-j\omega)t}}{s - j\omega} - \frac{e^{-j\theta} \cdot e^{-(s+j\omega)t}}{s + j\omega} \right]_0^\infty \\
&= \frac{1}{2} \left(\frac{e^{j\theta}}{s - j\omega} + \frac{e^{-j\theta}}{s + j\omega} \right) = \frac{1}{2} \frac{s \left(e^{j\theta} + e^{-j\theta} \right) + j\omega \left(e^{j\theta} - e^{-j\theta} \right)}{s^2 + \omega^2} \\
&= \frac{s \cdot \cos\theta - \omega \cdot \sin\theta}{s^2 + \omega^2}
\end{aligned}
$$

(4)

$$
F_4(s) = \frac{6}{(s+2)^2 + 9} = \frac{6}{(s + 2 + j3)(s + 2 - j3)}
$$

$$
\begin{aligned}
f_4(t) &= (s+2-j3)F_4(s)e^{st}\Big|_{s=-2+j3} + (s+2+j3)F_4(s)e^{st}\Big|_{s=-2-j3} \\
&= \left.\frac{6e^{st}}{s+2+j3}\right|_{s=-2+j3} + \left.\frac{6e^{st}}{s+2-j3}\right|_{s=-2-j3} = \frac{1}{j}e^{(-2+j3)t} - \frac{1}{j}e^{(-2-j3)t} \\
&= e^{-2t}\cdot\frac{\left(e^{j3t}-e^{-j3t}\right)}{j} = 2e^{-2t}\sin 3t
\end{aligned}
$$

または，ラプラス変換表と推移定理を利用して

$$F_4(s) = \frac{2\cdot 3}{(s+2)^2+3^2}$$

より

$$f_4(t) = 2e^{-2t}\sin 3t$$

(5)

$$F_5(s) = \frac{1}{s^3+3s^2} = \frac{1}{s^2(s+3)}$$

$$
\begin{aligned}
f_5(t) &= \frac{1}{(2-1)!}\frac{d}{ds}\left(s^2F_5(s)e^{st}\right)\Big|_{s=0} + (s+3)F_5(s)e^{st}\Big|_{s=-3} = \frac{d}{ds}\left(\frac{e^{st}}{s+3}\right)\Big|_{s=0} + \left.\frac{e^{st}}{s^2}\right|_{s=-3} \\
&= \left\{\frac{te^{st}}{s+3} - \frac{e^{st}}{(s+3)^2}\right\}\Big|_{s=0} + \left.\frac{e^{st}}{s^2}\right|_{s=-3} = \frac{t}{3} - \frac{1}{9} + \frac{e^{-3t}}{9} = \frac{3t-1+e^{-3t}}{9}
\end{aligned}
$$

または

$$F_5(s) = \frac{1}{s^3+3s^2} = \frac{1}{s^2(s+3)} = \frac{K_1s+K_2}{s^2} + \frac{K_3}{s+3}$$

と置くと，係数比較により

$$
\begin{cases}
K_1 + K_3 = 0 \\
3K_1 + K_2 = 0 \\
3K_2 = 1
\end{cases}
\quad\rightarrow\quad K_2 = \frac{1}{3},\quad K_1 = -\frac{K_2}{3} = -\frac{1}{9},\quad K_3 = -K_1 = \frac{1}{9}
$$

と求まるので

$$F_5(s) = \frac{1}{9}\left(-\frac{1}{s} + \frac{3}{s^2} + \frac{1}{s+3}\right)$$

ラプラス変換表より

$$f_5(t) = \frac{-1+3t+e^{-3t}}{9}$$