小テスト

(1)，(2)，(3) のラプラス変換と (4)，(5) のラプラス逆変換を求めよ．

$$(1)\ f_1(t) = e^{\beta t} \quad (2)\ f_2(t) = \sinh \beta t = \frac{e^{\beta t} - e^{-\beta t}}{2}, \quad (3)\ f_3(t) = \begin{cases} 1 & (0 \le t \le T) \\ 0 & (t < 0, t > T) \end{cases}$$

$$(4)\ F_4(s) = \frac{1}{(s+1)(s+2)}, \quad (5)\ F_5(s) = \frac{2}{(s+1)^2 - 2^2}$$

解答

(1)

$$F_1(s) = \int_0^\infty f_1(t)e^{-st}dt = \int_0^\infty e^{-(s-\beta)t}dt = \left[-\frac{e^{-(s-\beta)t}}{(s-\beta)}\right]_0^\infty = \frac{1}{s-\beta}$$

(2)

$$\begin{aligned} F_2(s) &= \int_0^\infty f_2(t)e^{-st}dt = \frac{1}{2}\int_0^\infty \left(e^{-(s-\beta)t} - e^{-(s+\beta)t}\right)dt \quad \left(= \frac{1}{2}\left[-\frac{e^{-(s-\beta)t}}{(s-\beta)} + \frac{e^{-(s+\beta)t}}{(s+\beta)}\right]_0^\infty\right) \\ &= \frac{1}{2}\left(\frac{1}{s-\beta} - \frac{1}{s+\beta}\right) = \frac{1}{2}\frac{(s+\beta)-(s-\beta)}{s^2-\beta^2} = \frac{\beta}{s^2-\beta^2} \end{aligned}$$

(3)

$$F_3(s) = \int_0^\infty f_3(t)e^{-st} = \int_0^T 1 \cdot e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^T = \frac{1-e^{-Ts}}{s}$$

(4) 留数演算から求めると

$$\begin{aligned} f_4(t) &= (s+1)F_4(s)e^{st}\Big|_{s=-1} + (s+2)F_4(s)e^{st}\Big|_{s=-2} = \left.\frac{1}{s+2}\right|_{s=-1} e^{-t} + \left.\frac{1}{s+1}\right|_{s=-2} e^{-2t} \\ &= e^{-t} - e^{-2t} \end{aligned}$$

(5) (2) のラプラス変換式で $\beta = 1$ とすると

$$\mathscr{L}\{\sinh 2t\} = \frac{2}{s^2-2^2} \to \mathscr{L}^{-1}\left\{\frac{2}{s^2-2^2}\right\} = \sinh 2t$$

推移定理を用いて $s \to s+1$ とすると

$$\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2-2^2}\right\} = e^{-t}\sinh 2t$$

したがて

$$f_5(t) = e^{-t}\sinh 2t$$