複エネルギー回路の過渡現象

RLC 直列回路

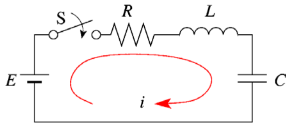
回路方程式(閉路方程式)は

$$v_L + v_R + v_C = L\frac{di}{dt} + Ri + \frac{q}{C} = E$$

$$i = \frac{dq}{dt}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$

$$q = q_s + q_t$$



定常解

$$\frac{q_s}{C} = E \longrightarrow q_s = CE$$

過渡解

$$L\frac{d^2q_t}{dt^2} + R\frac{dq_t}{dt} + \frac{q_t}{C} = 0 \longrightarrow q_t = Ae^{mt}$$

$$\frac{dq_t}{dt} = mAe^{mt} = mq_t$$

$$\left(Lm^2 + Rm + \frac{1}{C}\right)q_t = 0$$

$$\longrightarrow m = \frac{-R \pm \sqrt{R^2 - 4(L/C)}}{2L} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\left(\alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}\right)$$

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(1) $\alpha > \omega_0$ $\left(\beta = \sqrt{\alpha^2 - \omega_0^2} \rightarrow \omega_0 = \sqrt{\alpha^2 - \beta^2} = 1/\sqrt{LC}\right)$

$$q = q_s + q_t = CE + A_1 e^{(-\alpha+\beta)t} + A_2 e^{(-\alpha-\beta)t} = CE + e^{-\alpha t}\left(A_1 e^{\beta t} + A_2 e^{-\beta t}\right)$$

$$= CE + e^{-\alpha t}\left(B_1 \cosh \beta t + B_2 \sinh \beta t\right)$$

$$i = -\alpha e^{-\alpha t}\left(B_1 \cosh \beta t + B_2 \sinh \beta t\right) + \beta e^{-\alpha t}\left(B_1 \sinh \beta t + B_2 \cosh \beta t\right)$$

$$= e^{-\alpha t}\left[(-\alpha B_1 + \beta B_2)\cosh \beta t + (-\alpha B_2 + \beta B_1)\sinh \beta t\right]$$

初期条件 $t = 0$ で $q = 0, \quad i = 0$

$$q(0) = CE + B_1 = 0 \longrightarrow B_1 = -CE$$

$$i(0) = -\alpha B_1 + \beta B_2 = 0 \longrightarrow B_2 = \frac{\alpha}{\beta}B_1 = -CE\frac{\alpha}{\beta}$$

以上より

$$\phi = \tanh^{-1}\frac{\beta}{\alpha}$$

$$q = CE\left[1 - e^{-\alpha t}\left(\cosh \beta t + \frac{\alpha}{\beta}\sinh \beta t\right)\right]$$

$$= CE\left[1 - \sqrt{\left(\frac{\alpha}{\beta}\right)^2 - 1}\, e^{-\alpha t}\sinh(\beta t + \phi)\right] = CE\left[1 - \frac{\omega_0}{\beta}e^{-\alpha t}\sinh(\beta t + \phi)\right]$$

$$i = CE\frac{\alpha^2 - \beta^2}{\beta}e^{-\alpha t}\sinh(\beta t + \phi) = \frac{E}{\beta L}e^{-\alpha t}\sinh(\beta t + \phi)$$

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(3) $\alpha < \omega_0$ $\left(\beta = j\omega = j\sqrt{\omega_0^2 - \alpha^2} \rightarrow \omega_0 = \sqrt{\alpha^2 + \omega^2} = 1/\sqrt{LC}\right)$

$$q = q_s + q_t = CE + A_1 e^{(-\alpha+j\omega)t} + A_2 e^{(-\alpha-j\omega)t} = CE + e^{-\alpha t}\left(A_1 e^{j\omega t} + A_2 e^{-j\omega t}\right)$$

$$= CE + e^{-\alpha t}\left(B_1 \cos \omega t + B_2 \sin \omega t\right)$$

$$i = -\alpha e^{-\alpha t}\left(B_1 \cos \omega t + B_2 \sin \omega t\right) + \omega e^{-\alpha t}\left(-B_1 \sin \omega t + B_2 \cos \omega t\right)$$

$$= e^{-\alpha t}\left[(-\alpha B_1 + \omega B_2)\cos \omega t + (-\alpha B_2 - \omega B_1)\sin \omega t\right]$$

初期条件 $t = 0$ で $q = 0, \quad i = 0$

$$q(0) = CE + B_1 = 0 \longrightarrow B_1 = -CE$$

$$i(0) = -\alpha B_1 + \omega B_2 = 0 \longrightarrow B_2 = \frac{\alpha}{\omega}B_1 = -CE\frac{\alpha}{\omega}$$

以上より

$$\phi = \tan^{-1}\frac{\omega}{\alpha}$$

$$q = CE\left[1 - e^{-\alpha t}\left(\cos \omega t + \frac{\alpha}{\omega}\sin \omega t\right)\right]$$

$$= CE\left[1 - \sqrt{\left(\frac{\alpha}{\omega}\right)^2 + 1}\cdot e^{-\alpha t}\sin(\omega t + \phi)\right] = CE\left[1 - \frac{\omega_0}{\omega}e^{-\alpha t}\sin(\omega t + \phi)\right]$$

$$i = CE\frac{\alpha^2 + \omega^2}{\omega}e^{-\alpha t}\sin(\omega t + \phi) = \frac{E}{\omega L}e^{-\alpha t}\sin(\omega t + \phi)$$

三角関数と双曲線関数

$$\cos\omega t = \frac{e^{j\omega t}+e^{-j\omega t}}{2} \qquad \cosh\beta t = \frac{e^{\beta t}+e^{-\beta t}}{2}$$

$$\sin\omega t = \frac{e^{j\omega t}-e^{-j\omega t}}{j2} \qquad \sinh\beta t = \frac{e^{\beta t}-e^{-\beta t}}{2}$$

$\omega = -j\beta$

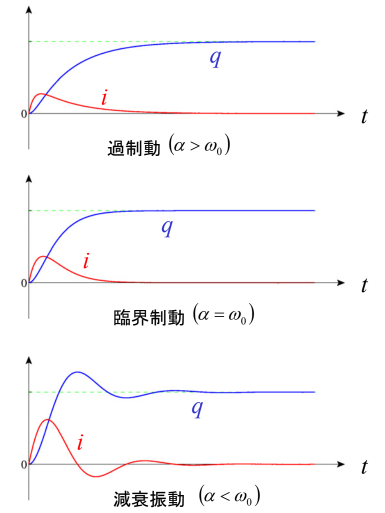
$$\cos\omega t = \frac{e^{\beta t}+e^{-\beta t}}{2} = \cosh\beta t$$

$$\sin\omega t = \frac{e^{\beta t}-e^{-\beta t}}{j2} = -j\sinh\beta t$$

$\beta = j\omega$

$$\cosh\beta t = \frac{e^{j\omega t}+e^{-j\omega t}}{2} = \cos\omega t$$

$$\sinh\beta t = \frac{e^{j\omega t}-e^{-j\omega t}}{2} = j\sin\omega t$$



(2) $\alpha = \omega_0 \longrightarrow m = -\alpha$

$$q = q_s + q_t = CE + (A_1 + A_2 t)e^{-\alpha t}$$

$$i = \frac{dq}{dt} = A_2 e^{-\alpha t} - \alpha(A_1 + A_2 t)e^{-\alpha t} = \{(A_2 - \alpha A_1) - \alpha A_2 t\}e^{-\alpha t}$$

初期条件 $t=0$ で $q=0,\quad i=0$

$$q(0) = CE + A_1 = 0 \longrightarrow A_1 = -CE$$

$$i(0) = A_2 - \alpha A_1 = 0 \longrightarrow A_2 = \alpha A_1 = -\alpha CE$$

以上より

$$q = CE\left[1-(1+\alpha t)e^{-\alpha t}\right]$$

$$i = \alpha^2 CEte^{-\alpha t} = \omega_0^2 CEte^{-\alpha t} = \frac{E}{L}te^{-\alpha t}$$