On generalised framed surfaces in the Euclidean space

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Abstract

We have introduced framed surfaces as smooth surfaces with singular points. The framed surface is a surface with a moving frame based on the unit normal vector of the surface. Thus, the notion of framed surfaces (respectively, framed base surfaces) is locally equivalent to the notion of Legendre surfaces (respectively, frontals). A more general notion of singular surfaces, called generalised framed surfaces, is introduced in this talk. The notion of generalised framed surfaces includes not only the notion of framed surfaces, but also the notion of one-parameter families of framed curves. It also includes surfaces with corank one singularities. We investigate properties of generalised framed surfaces.

1 Definition

1.1 Framed surfaces

We quickly review the theory of framed surfaces in the Euclidean 3-space, in detail see [5, 6]. Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a smooth mapping.

Definition 1.1 We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a *framed surface* if $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) = \boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) = 0$ for all $(u,v) \in U$, where $\boldsymbol{x}_u(u,v) = (\partial \boldsymbol{x}/\partial u)(u,v)$ and $\boldsymbol{x}_v(u,v) = (\partial \boldsymbol{x}/\partial v)(u,v)$. We say that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a *framed base surface* if there exists $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface.

By definition, the framed base surface is a frontal. On the other hand, the frontal is a framed base surface at least locally.

1.2 One-parameter families of framed curves

We also review the theory of one-parameter families of framed curves in the Euclidean 3-space, in detail see [6, 15]. Let $(\gamma, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a smooth mapping.

Definition 1.2 We say that $(\gamma, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ is a *one-parameter family of framed curves with respect to* u (respectively, *with respect to* v) if $(\gamma(\cdot, v), \nu_1(\cdot, v), \nu_2(\cdot, v))$ is a framed curve for each v (respectively, $(\gamma(u, \cdot), \nu_1(u, \cdot), \nu_2(u, \cdot))$ is a framed curve for each u), that is, $\gamma_u(u,v) \cdot \nu_1(u,v) = \gamma_u(u,v) \cdot \nu_2(u,v) = 0$ (respectively, $\gamma_v(u,v) \cdot \nu_1(u,v) = \gamma_v(u,v) \cdot \nu_2(u,v) = 0$) for all $(u,v) \in U$. We say that γ is a *one-parameter family of framed base curves with respect to* u (respectively, *with respect to* v) if there exists $(\nu_1, \nu_2) : U \to \Delta$ such that (γ, ν_1, ν_2) is a one-parameter family of framed curves with respect to u (respectively, *with respect to* v).

1.3 Generalised framed surfaces

We give a definition of a generalisation of the framed surfaces and one-parameter families of framed curves. Let $(\boldsymbol{x},\nu_1,\nu_2):U\to\mathbb{R}^3\times\Delta$ be a smooth mapping. We denote $\nu=\boldsymbol{x}_u\times\boldsymbol{x}_v$.

Definition 1.3 We say that $(\boldsymbol{x},\nu_1,\nu_2):U\to\mathbb{R}^3\times\Delta$ is a *generalised framed surface* if there exist smooth functions $\alpha,\beta:U\to\mathbb{R}$ such that $\nu(u,v)=\alpha(u,v)\nu_1(u,v)+\beta(u,v)\nu_2(u,v)$ for all $(u,v)\in U$. We say that $\boldsymbol{x}:U\to\mathbb{R}^3$ is a *generalised framed base surface* if there exists $(\nu_1,\nu_2):U\to\Delta$ such that $(\boldsymbol{x},\nu_1,\nu_2)$ is a generalised framed surface.

Remark 1.4 Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}):U\to\mathbb{R}^3\times\Delta$ be a framed surface with basic invariants $(\mathcal{G},\mathcal{F}_1,\mathcal{F}_2)$. Then $\nu(u,v)=\boldsymbol{x}_u(u,v)\times\boldsymbol{x}_v(u,v)=(a_1(u,v)b_2(u,v)-a_2(u,v)b_1(u,v))\boldsymbol{n}(u,v)$. If we take $\alpha(u,v)=a_1(u,v)b_2(u,v)-a_2(u,v)b_1(u,v)$ and $\beta(u,v)=0$, then $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ is also a generalised framed surface.

Remark 1.5 Let $(\gamma,\nu_1,\nu_2):U\to\mathbb{R}^3\times\Delta$ be a one-parameter family of framed curves with respect to u with curvature (ℓ,m,n,r,L,M,N,P,Q,R). Then $\nu(u,v)=\gamma_u(u,v)\times\gamma_v(u,v)=-r(u,v)Q(u,v)\nu_1(u,v)+r(u,v)P(u,v)\nu_2(u,v)$. If we take $\alpha(u,v)=-r(u,v)Q(u,v)$ and $\beta(u,v)=r(u,v)P(u,v)$, then (γ,ν_1,ν_2) is also a generalised framed surface.

We denote $\nu_3(u,v)=\nu_1(u,v)\times\nu_2(u,v)$. Then $\{\nu_1(u,v),\nu_2(u,v),\nu_3(u,v)\}$ is a moving frame along $\boldsymbol{x}(u,v)$ and we have the following systems of differential equations:

$$\begin{pmatrix}\boldsymbol{x}_u\\ \boldsymbol{x}_v\end{pmatrix}=\begin{pmatrix}a_1&b_1&c_1\\ a_2&b_2&c_2\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix},$$

$$\begin{pmatrix}\nu_{1u}\\ \nu_{2u}\\ \nu_{3u}\end{pmatrix}=\begin{pmatrix}0&e_1&f_1\\ -e_1&0&g_1\\ -f_1&-g_1&0\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix},\quad \begin{pmatrix}\nu_{1v}\\ \nu_{2v}\\ \nu_{3v}\end{pmatrix}=\begin{pmatrix}0&e_2&f_2\\ -e_2&0&g_2\\ -f_2&-g_2&0\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix},$$

where $a_i,b_i,c_i,e_i,f_i,g_i:U\to\mathbb{R}, i=1,2$ are smooth functions with $a_1b_2-a_2b_1=0$. We call the functions *basic invariants* of the generalised framed surface. We denote the above matrices by $\mathcal{G},\mathcal{F}_1,\mathcal{F}_2$, respectively. We also call the matrices $(\mathcal{G},\mathcal{F}_1,\mathcal{F}_2)$ *basic invariants* of the generalised framed surface $(\boldsymbol{x},\nu_1,\nu_2)$. By definition, we have

$$\alpha(u,v)=\det\begin{pmatrix}b_1(u,v)&c_1(u,v)\\ b_2(u,v)&c_2(u,v)\end{pmatrix},\ \beta(u,v)=-\det\begin{pmatrix}a_1(u,v)&c_1(u,v)\\ a_2(u,v)&c_2(u,v)\end{pmatrix}.$$

Since the integrability conditions $\boldsymbol{x}_{uv}=\boldsymbol{x}_{vu}$ and $\mathcal{F}_{2u}-\mathcal{F}_{1v}=\mathcal{F}_1\mathcal{F}_2-\mathcal{F}_2\mathcal{F}_1$, the basic invariants should be satisfied the following conditions:

$$\begin{cases}a_{1v}-b_1e_2-c_1f_2=a_{2u}-b_2e_1-c_2f_1,\\ b_{1v}+a_1e_2-c_1g_2=b_{2u}+a_2e_1-c_2g_1,\\ c_{1v}+a_1f_2+b_1g_2=c_{2u}+a_2f_1+b_2g_1,\end{cases}\tag{1}$$

$$\begin{cases}e_{1v}-f_1g_2=e_{2u}-f_2g_1,\\ f_{1v}-e_2g_1=f_{2u}-e_1g_2,\\ g_{1v}-e_1f_2=g_{2u}-e_2f_1.\end{cases}\tag{2}$$

2 Main results

We give fundamental theorems for generalised framed surfaces, that is, the existence and uniqueness theorems for the basic invariants of generalised framed surfaces.

Theorem 2.1 (Existence Theorem for generalised framed surfaces) *Let* $(a_i, b_i, c_i, e_i, f_i, g_i) : I \to \mathbb{R}^{12}, i = 1, 2$ *be a smooth mapping satisfying* $a_1 b_2 - a_2 b_1 = 0$*, the integrability conditions* (1) *and* (2)*. Then there exists a generalised framed surface* $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ *whose associated basic invariants are* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$*.*

Definition 2.2 Let $(\boldsymbol{x}, \nu_1, \nu_2), (\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ be generalised framed surfaces. We say that $(\boldsymbol{x}, \nu_1, \nu_2)$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are *congruent as generalised framed surfaces* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that $\widetilde{\boldsymbol{x}}(u, v) = A(\boldsymbol{x}(u, v)) + \boldsymbol{a}$, $\widetilde{\nu}_1(u, v) = A(\nu_1(u, v))$ and $\widetilde{\nu}_2(u, v) = A(\nu_2(u, v))$ for all $(u, v) \in U$.

Theorem 2.3 (Uniqueness Theorem for generalised framed surfaces) *Let* $(\boldsymbol{x}, \nu_1, \nu_2)$*,* $(\widetilde{\boldsymbol{x}}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ *be generalised framed surfaces with basic invariants* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$*,* $(\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$*, respectively. Then* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *and* $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ *are congruent as generalised framed surfaces if and only if the basic invariants* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ *and* $(\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$ *coincide.*

We give a condition for a surface to become a generalised framed base surface.

Theorem 2.4 *Let* $\boldsymbol{x} : U \to \mathbb{R}^3$ *be a smooth mapping. We denote* $\nu = \boldsymbol{x}_u \times \boldsymbol{x}_v = p_1 \boldsymbol{e}_1 + p_2 \boldsymbol{e}_2 + p_3 \boldsymbol{e}_3$*, where* $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ *are the canonical basis. Then* $\boldsymbol{x}$ *is a generalised framed base surface at least locally if and only if the functions* p_1, p_2, p_3 *are linearly dependent.*

Theorem 2.5 *Let* $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ *be a generalised framed surface with* $\nu = \alpha \nu_1 + \beta \nu_2$*. Then* $\boldsymbol{x}$ *is a framed base surface at least locally if and only if the functions* α *and* β *are linearly dependent.*

2.1 Corank one singularities

Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a smooth mapping. Suppose that $\mathrm{corank}(d\boldsymbol{x}) = 1$ at a point $p \in U$. By using a parameter change of U, we may assume that $\boldsymbol{x}$ is given by $\boldsymbol{x}(u, v) = (u, f(u, v), g(u, v))$ at least locally, where $f, g : U \to \mathbb{R}$ are smooth functions. Then corank one singularities are always generalised framed base surfaces at least locally.

Theorem 2.6 *Suppose that* $\boldsymbol{x} : U \to \mathbb{R}^3$ *is given by* $\boldsymbol{x}(u, v) = (u, f(u, v), g(u, v))$*. Then* $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ *is a generalised framed surface, where*

$$\nu_1(u, v) = \frac{(f_u(u, v), -1, 0)}{\sqrt{1 + f_u(u, v)^2}},$$

$$\nu_2(u, v) = \frac{(g_u(u, v), f_u(u, v) g_u(u, v), -f_u(u, v)^2 - 1)}{\sqrt{1 + f_u(u, v)^2} \sqrt{1 + f_u(u, v)^2 + g_u(u, v)^2}}$$

with

$$\alpha(u,v) = \frac{(1+f_u(u,v)^2)g_v(u,v) - f_u(u,v)f_v(u,v)g_u(u,v)}{\sqrt{1+f_u(u,v)^2}},$$
$$\beta(u,v) = -\frac{f_v(u,v)\sqrt{1+f_u(u,v)^2+g_u(u,v)^2}}{\sqrt{1+f_u(u,v)^2}}.$$

2.2 Corank two singularities

Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a smooth mapping. Suppose that $\mathrm{corank}(d\boldsymbol{x}) = 2$ at a point $p \in U$. We consider one of the components of $\boldsymbol{x}(u,v)$ is 2-jet, that is, $\boldsymbol{x}(u,v)$ is given by

$$(i)\ \left(\frac{1}{2}(u^2+v^2), f(u,v), g(u,v)\right),$$
$$(ii)\ \left(\frac{1}{2}(u^2-v^2), f(u,v), g(u,v)\right),$$
$$(iii)\ \left(\frac{1}{2}u^2, f(u,v), g(u,v)\right),$$

by using parameter change and up to sign, where $f, g : U \to \mathbb{R}$ are smooth functions. By a direct calculation, $\nu(u,v)$ is given by

(i) $(f_u(u,v)g_v(u,v) - f_v(u,v)g_u(u,v), -(ug_v(u,v) - vg_u(u,v)), uf_v(u,v) - vf_u(u,v))$,
(ii) $(f_u(u,v)g_v(u,v) - f_v(u,v)g_u(u,v), -(ug_v(u,v) + vg_u(u,v)), uf_v(u,v) + vf_u(u,v))$,
(iii) $(f_u(u,v)g_v(u,v) - f_v(u,v)g_u(u,v), -ug_v(u,v), uf_v(u,v))$,

respectively. By Theorem 2.4, $\boldsymbol{x}$ is a generalised framed base surface at least locally if and only if the components of $\nu(u,v)$ are linearly dependent.

As special cases, we consider two of components of $\boldsymbol{x}(u,v)$ are 2-jet.

Proposition 2.7 *Suppose that* $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ *is given by*

$$\boldsymbol{x}(u,v) = \left(\frac{1}{2}(u^2+v^2), \frac{1}{2}(u^2-v^2), g(u,v)\right)$$

and $j^2g(0) = 0$*. Then we have the following.*

(1) $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ *is a generalised framed base surface germ if and only if there exists a function* $h : (\mathbb{R}^2, 0) \to \mathbb{R}$ *such that* $g_u = uh$ *or* $g_v = vh$*.*

(2) *Suppose that* $\boldsymbol{x}$ *is a generalised framed base surface germ. Then* $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ *is a framed base surface germ if and only if there exist functions* $h_1, h_2 : (\mathbb{R}^2, 0) \to \mathbb{R}$ *such that* $g_u = uh_1$ *and* $g_v = vh_2$*.*

Proposition 2.8 *Suppose that* $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ *is given by*

$$\boldsymbol{x}(u,v) = \left(\frac{1}{2}u^2, \frac{1}{2}v^2, g(u,v)\right)$$

and $j^2g(0) = 0$*. Then we have the following.*

(1) $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ *is a generalised framed base surface germ if and only if there exists a function* $h : (\mathbb{R}^2, 0) \to \mathbb{R}$ *such that* $g_u = uh$ *or* $g_v = vh$.

(2) *Suppose that* $\boldsymbol{x}$ *is a generalised framed base surface germ. Then* $\boldsymbol{x} : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ *is a framed base surface germ if and only if there exist functions* $h_1, h_2 : (\mathbb{R}^2, 0) \to \mathbb{R}$ *such that* $g_u = uh_1$ *and* $g_v = vh_2$.

3 Application

Let $(\boldsymbol{x}, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a generalised framed surface with $\nu = \alpha\nu_1 + \beta\nu_2$ and basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. We consider parallel surfaces of the generalised framed surface $(\boldsymbol{x}, \nu_1, \nu_2)$.

Definition 3.1 We say that $\boldsymbol{x}^\lambda : U \to \mathbb{R}^3$, $\boldsymbol{x}^\lambda = \boldsymbol{x} + \lambda\nu$ is a *parallel surface* of the generalised framed surface $(\boldsymbol{x}, \nu_1, \nu_2)$, where λ is a non-zero constant.

Proposition 3.2 *Let* $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, p) \to \mathbb{R}^3 \times \Delta$ *be a generalised framed surface with* $k\widetilde{\nu} = k\widetilde{\alpha}\nu_1 + k\widetilde{\beta}\nu_2$ *and basic invariants* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. *Suppose that* $(\widetilde{\alpha}, \widetilde{\beta})(p) = 0$. *Then we have the following.*

(1) *If* $(a_1, a_2)(p) = (b_1, b_2)(p) = 0$ *and* $(\widetilde{\alpha}_u, \widetilde{\alpha}_v, \widetilde{\beta}_u, \widetilde{\beta}_v)(p) \neq 0$, *then* $\boldsymbol{x}^\lambda[k]$ *is also a generalised framed base surface around* p.

(2) *If* $(c_1, c_2)(p) \neq 0$, *then* $\boldsymbol{x}^\lambda[k]$ *is also a generalised framed base surface around* p.

We define the other type of parallel surfaces of the generalised framed surface $(\boldsymbol{x}, \nu_1, \nu_2)$.

Definition 3.3 We say that $\boldsymbol{x}^\lambda[\theta] : U \to \mathbb{R}^3$, $\boldsymbol{x}^\lambda[\theta] = \boldsymbol{x} + \lambda(\cos\theta\nu_1 + \sin\theta\nu_2)$ is a θ*-parallel surface* of the generalised framed surface $(\boldsymbol{x}, \nu_1, \nu_2)$, where λ is a non-zero constant and θ is a constant.

Proposition 3.4 *Let* $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ *be a generalised framed surface which is given by the form of Theorem 2.6. Suppose that* $f_u(0) = f_v(0) = g_u(0) = g_v(0) = 0$. *Then we have the following.*

(1) *If* $f_{uu}(0)\cos\theta + g_{uu}(0)\sin\theta = 0$, *then* $\boldsymbol{x}^\lambda[\theta]$ *is a generalised framed base surface germ.*

(2) *If* $f_{uv}(0)\cos\theta + g_{uv}(0)\sin\theta \neq 0$, *then* $\boldsymbol{x}^\lambda[\theta]$ *is a generalised framed base surface germ.*

4 Example

Example 4.1 (Cross cap) Let $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$,

$$\boldsymbol{x}(u, v) = (u, v^2, uv),\ \nu_1(u, v) = (0, -1, 0),\ \nu_2(u, v) = \frac{1}{\sqrt{1+v^2}}(v, 0, -1).$$

Then $(\boldsymbol{x}, \nu_1, \nu_2)$ is a generalised framed surface germ with $\alpha(u, v) = u, \beta(u, v) = -2v\sqrt{1+v^2}$.

Example 4.2 (H_k singular point) Let $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ be

$$\boldsymbol{x}(u, v) = (u, uv + v^{3k-1}, v^3),\ \nu_1(u, v) = \frac{(-v, 1, 0)}{\sqrt{1+v^2}},\ \nu_2(u, v) = (0, 0, 1),$$

where k is a natural number with $k \geq 2$. Note that 0 is a H_k singular point of $\boldsymbol{x}$ (cf. [14]). Then $(\boldsymbol{x}, \nu_1, \nu_2)$ is a generalised framed surface germ with

$$\alpha(u,v) = -3v^2\sqrt{1+v^2},\ \beta(u,v) = u + (3k-1)v^{3k-2}.$$

Example 4.3 Let $(\boldsymbol{x}, \nu_1, \nu_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ be

$$\boldsymbol{x}(u,v) = \left(\frac{1}{2}u^2, \frac{1}{2}v^2, u^{k+2}v\right),\ \nu_1(u,v) = (0,-1,0),\ \nu_2(u,v) = \frac{((k+2)u^k v, 0, -1)}{\sqrt{(k+2)^2u^{2k}v^2+1}},$$

where k is a natural number. Note that 0 is a corank two singular point of $\boldsymbol{x}$. By $g(u,v) = u^{k+2}v$ in Proposition 2.8 (1), $(\boldsymbol{x}, \nu_1, \nu_2)$ is a generalised framed surface germ with

$$\alpha(u,v) = -u^{k+3},\ \beta(u,v) = -uv\sqrt{(k+2)^2u^{2k}v^2+1}.$$

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