Eisenstein series

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Eisenstein series are a very important subject in the theory of automorphic forms. We will introduce here the notion of holomorphic Eisenstein series in one variable on the full modular group and discuss some of their basic properties. Then we will describe an application in the theory of quadratic forms, regarding representations of positive integers as sums of integral squares.

1. Basic definitions and properties of Eisenstein series

We denote by $\mathcal{H}$ the complex upper half-plane consisting of complex numbers with positive imaginary part. Recall that $\Gamma_1 := SL_2(\mathbf{Z})$ operates on $\mathcal{H}$ in the usual way by fractional linear transformations. Fix a positive integer $k \geq 4$ which is even.

Definition. *The series*

$$G_k(z) := \sum_{m,n}^{*} (mz+n)^{-k} \qquad (z \in \mathcal{H})$$

(summation over all pairs of integers with $(0,0)$ *excluded) is called an Eisenstein series of weight* k *on* Γ_1*.*

Theorem 1. i) The series is absolutely uniformly convergent on subsets $D_\epsilon := \{z = x+iy \in \mathcal{H} \,|\, y \geq \epsilon, |x| \leq \epsilon^{-1}\}$, for any $\epsilon > 0$. In particular, $G_k(z)$ is a holomorphic function.

ii) One has the transformation formula

$$G_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^k G_k(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$.

iii) The Fourier expansion

$$G_k(z) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n \qquad (q := e^{2\pi i z})$$

holds, where $\zeta(k)$ is the value of the Riemann zeta function at $s = k$ and $\sigma_{k-1}(n) := \sum_{d|n, d>0} d^{k-1}$.

Thus G_k is an element of $M_k(1)$, the space of modular forms of weight k on Γ_1 consisting of complex-valued holomorphic functions f on $\mathcal{H}$, which satisfy the transformation law $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$ and have a Fourier expansion $f(z) = \sum_{n\geq 0} a(n)q^n$.

Theorem 2. One has $M_k(1) = \mathbf{C}G_k \oplus S_k(1)$ where $S_k(1)$ is the subspace of cusp forms (require $a(0) = 0$).

Definition. *We define the normalized Eisenstein series by*

$$E_k(z) := \frac{1}{2\zeta(k)} G_k(z) \qquad (z \in \mathcal{H}).$$

One shows that

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n,$$

where B_k is th k-th Bernoulli number. In particular one has

$$E_4(z) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n, \quad E_6(z) = 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n.$$

Theorem 3. *The following assertions hold:*

i) $dim M_k(1) < \infty$;

ii) $M_k(1) = \oplus_{a,b\geq 0, 4a+6b=k} \mathbf{C} E_4^a E_6^b$.

Although assertion ii) from an algebraic point of view is very satisfying, the Fourier coefficients of the basis elements given are complicated ("many multiplications"), hence are not adequate to produce any "simple" formulas for the Fourier coefficients of any given $f \in M_k(1)$. The question arises if one can do better. The answer is often "yes", also in more general cases.

We will shortly describe a special case in the next section (where a "simple" basis function is just the cuspidal part of the product of *two* Eisenstein series). We will also give an application regarding sums of integral squares.

We will start with recalling rsults on the representation of positive integers as sums of integral squares.

2. Sums of integral squares

For a positive integer s divisible by 4 and for a positive integer n we let $r_s(n)$ be the number of representations of n as a sum of s integral squares. What one wants are "nice" and finite formulas for $r_s(n)$, and this had been the subject of many investigations over the past decades and centuries.

One idea (going back to Jacobi) is to use generating series. So we define the "basic" theta function

$$\theta(z) := \sum_{n\in\mathbf{Z}} q^{n^2} \qquad (z \in \mathcal{H}).$$

Then

$$\theta^s(z) = 1 + \sum_{n\geq 1} r_s(n) q^n.$$

It is a non-trivial fact that θ^s is a modular form of weight $s/2$ of level 4, i.e. has a corresponding transformation formula under matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$ with c divisible by 4.

The following has been well-known:

i) For small s (i.e. $s = 4, 8$), $r_s(n)$ is an elementary modified divisor sum, e.g.

$$r_4(n) = \sum_{d|n}^{*} d$$

where the summation runs over all positive divisors of d of n not divisble by 4 (Lagrange, Jacobi).

ii) If $s > 8$, then $\theta^s = E + f$, where E is an Eisenstein series of level 4 and f is a *non-zero* cusp form (Rankin, 1965).

Around 2001, Milne and independently Zagier obtained formulas for $r_s(n)$ whenever $s = 4j^2$ or $s = 4j^2 + 4j$ with j a positive integer, in terms of explicit finite sums of products of j modified divisor sums.

Conjecture *(Chan-Chua, 2003) For* $8|s$, θ^s *is a unique linear combination of* $\frac{s}{8} - 1$ *products of two specific Eisenstein series of level 4.*

This conjecture was proved by K. Tasaka around 2013 using "double shuffle" relations for so-called double Eisenstein series and a result of O. Imamoglu and W. Kohnen (2005) on generators for the space of cusp forms of weight k and level 2, given by the cuspidal parts of products of two Eisenstein series. (The proof of the latter uses the so-called Rankin-Selberg method and Eichler-Shimura theory.)

Results of the latter type were also proved independently by Borisov-Gunnels, Yang-Fukuhara, Kilger, Khuri-Makdisi and Kohnen-Martin.

These results lead to explicit expressions for $r_s(n)$ as a linear combination of products of two divisor sums.