Extremal lattices and modular forms

(S.Böcherer)

Let (V,q) be an m-dimensional positive definite quadratic space over $\mathbb{Q}$ and L a $\mathbb{Z}$-lattice of full rank in V; we assume q to take even integral values on L and we will (after choosing a basis of L) identify L with an even integral matrix $Q = Q_L$). In a vague sense, L (or Q) will be called extremal if

$$\mathrm{Min}(L) := \inf\{q(\mathbf{x}) \mid \mathbf{x} \in L,\, \mathbf{x} \neq \mathbf{0}\}$$

is "as large as possible".
Of course, this has to be made more precise and there are several ways to do this, either geometrically or analytically via modular forms. We choose the latter approach and restrict ourselves to the case

$$m = 2k \;\text{ divisible by 4,} \qquad det(Q) \;\text{ is a perfect square}$$

Then, as is well known, the associated theta series is a modular form of weight k for the group $\Gamma_0(N)$ of "Haupttype":

$$\theta(L,\tau) = \sum_{\mathbf{x}\in L} e^{\pi i q(\mathbf{x})\tau} = \sum_{n=0}^{\infty} r_L(n) e^{2\pi i n\tau} \in M_k(N)$$

where N is the level of the lattice L, τ is an element of the complex upper half plane and the $r_L(n)$ denote the representation numbers

$$r_L(n) = \{\mathbf{x} \in L \mid q(\mathbf{x}) = 2n\}.$$

Assume now that $\theta(L)$ lies in a distinguished subspace $\mathcal{M} \subset M_k(N)$ which has the *"Weierstrass property"*; we define this notion (which is related to the notion of Weierstrass points on Riemann surfaces) as follows
The map

$$\begin{cases} \mathcal{M} & \longrightarrow & \mathbb{C}^{dim\mathcal{M}} \\ f(\tau) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n\tau} & \longmapsto & (a(0), a(1), \ldots, a(\dim(\mathcal{M}) - 1)) \end{cases}$$

is a bijection. It is clear how this definition should be modified if $\mathcal{M}$ contains by definition only modular forms with vanishing Fourier coefficient $a(0)$, e.g.

if $\mathcal{M}$ consists only of cusp forms.
Under such a condition, we call a lattice L "$\mathcal{M}$- extremal" iff

$$\theta(L,\tau) = 1 + \sum_{n=dim(\mathcal{M})}^{\infty} r_Q(n) e^{2\pi i n \tau}$$

or in other words, if Min(L)=2$dim(\mathcal{M})$.

The remarkable fact is that there may possibly be non-isometric lattices L_1 and L_2 which are $\mathcal{M}$-extremal, but their theta series have to be identical.
In general, we may say that the construction of such extreme lattices (or the proof of their non-existence) is an interesting difficult problems; the classification of extremal lattices may be easier than the classification of all lattices of some level or in some genus (in most cases, the classification of all lattices of some level is just impossible; the consideration of extremal lattices reduces the number of candidates considerably!).

Of course this notion only makes sense, if we know interesting classes of such distinguished subspaces $\mathcal{M}$, e.g. the full space $M_k(1)$ of modular forms of level 1 has the Weierstrass property and the well-known Leech lattice is then an $\mathcal{M} = M_{12}(1)$- extremal lattice.
In general, neither the spaces $M_k(N)$ nor their cuspidal subspaces have the Weierstrass property. In the case of squarefree level N Arakawa and I introduced an intermediate space
Definition:(N squarefree)

$$M_k(N)^* = \{f \in M_k(N) \ \mid \ \forall p \mid N : \ f \mid W_p^N + p^{1-\frac{k}{2}} f \mid U(p) = 0\}$$

where W_p^N and $U(p)$ are standard operators and we showed that this space always has the Weierstrass property.

After these preliminaries we come to the central question:

For which lattices L is $\theta(L) \in M_k(N)^*$???

If we can show that this is a reasonable class of lattices, then it makes sense to consider among these lattices the $M_k(N)^*$-extremal ones.
Our main result is the following:

Theorem: *Let L be an even integral lattice in a positive definite quadratic space (V,q) over $\mathbb{Q}$ of dimension $m = 2k$ divisible by 4. Assume that the discriminant of L is a square and that the (exact) level N of L is squarefree. Then*

$$\theta(L) \in M_k(N)^* \iff Ad(L) \text{ is a maximal lattice}$$

Here $Ad(L)$ denotes the lattice adjoint to L, its matrix is given by $N \cdot Q^{-1}$.

Now one can start to search for such "extremal lattices" among the lattices adjoint to maximal lattices.
For some examples see our paper (or look at a list of "interesting lattices", maintained by G.Nebe and J.Sloane, available on the web.)

The content of this talk was based mainly on my papers with T.Arakawa (J.Reine angew.Math.559, 2003) and with G.Nebe (J.Ramanujan Math.Soc.25, 2010)

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