On the decay property for the nonlinear Klein-Gordon equation in de Sitter spacetime

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1 Introduction

We consider two nonlinear problems for the wave equation. First we show the blow up result to the following initial value problem for the wave equation with weighted nonlinear terms in one space dimension:

$$\partial_t^2 u - \partial_x^2 u = G(x, u), \qquad (x, t) \in \mathbb{R} \times [0, \infty),$$
$$u(x, 0) = \varphi(x), \quad \partial_t u(x, 0) = \psi(x), \qquad x \in \mathbb{R},$$

where $\varphi(x) \in C^2(\mathbb{R})$, $\psi(x) \in C^1(\mathbb{R})$, and G is the nonlinear term such that $G(x, 0) = 0$ for $x \in \mathbb{R}$.

The second problem is the inital value problem for the nonlinear Klein-Gordon equation in the de Sitter spacetime,

$$\partial_t^2 \Phi + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = F(\Phi), \qquad (x, t) \in \mathbb{R}^n \times \mathbb{R},$$
$$\Phi(x, 0) = \varphi_0(x), \quad \partial_t \Phi(x, 0) = \varphi_1(x), \qquad x \in \mathbb{R}^n,$$

where $\varphi_0, \varphi_1 \in W^{N,2}(\mathbb{R}^n)$ and F is the nonlinear term. We derive pointwise decay estimates for the solution to the linear Klein-Gordon equation in the de Sitter spacetime with and without source term by using the decay estimate for the following linear wave equation with zero initial velocity and nonzero initial position,

$$v_{tt} - \Delta v = 0, \quad v(x, 0) = \varphi(x), \quad v_t(x, 0) = 0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}$$

where $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$. We use the estimates for proving that the inital value problem to the nonlinear Klein-Gordon equation admits a global solution for small initial data.

2 Wave equation with weighted nonlinear terms

We consider the initial value problem for nonlinear wave equations in one space dimension:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = G(x,u), & (x,t) \in \mathbb{R} \times [0,\infty), \\ u(x,0) = \varphi(x), \quad \partial_t u(x,0) = \psi(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $\varphi(x) \in C^2(\mathbb{R})$, $\psi(x) \in C^1(\mathbb{R})$, and the nonlinearity is typically given by

$$G(x,u) = \frac{|u|^{p-1} u}{(1+x^2)^{(a+1)/2}}, \tag{2.2}$$

where $p > 1$, $a \geq -1$. When $a = -1$, Kato [9] showed that for any $p > 1$, the solution for the problem blows up in finite time under certain positivity assumptions on initial data in one space dimension. In addition, Zhou [19] obtained upper and lower bounds of the lifespan in terms of the size of initial data in one space dimension. When $a > -1$, then the situation becomes different as was shown by Suzuki [14] in one space dimension. Indeed, when $p > (1+\sqrt{5})/2$ and $pa > 1$, the problem (2.1) has a global C^2-solution if φ and ψ are odd functions and their size are sufficiently small. On the other hand, when $-1 \leq a \leq 1$ and $p > 1$, the blow-up occurs. In addition, Kubo, Osaka and Yazici [10] obtained that when $p > 1$ and $pa > 1$, the solution exists globally if the initial data is sufficiently small and odd.

Our aim is this section to answer what will happen for the case of $p > 1$ and $a > 1$. We denote by T_* the lifespan of the C^2-solution to the problem (2.1), that is,

$$T_* = \sup \left\{ T \in (0,\infty) : \text{ (2.1) with (2.2) has a solution } u \in C^2(\mathbb{R} \times [0,T)) \right\}.$$

Now we are in a position to state our result.

Theorem 2.1. *Let $p > 1$ and $a \geq -1$. Assume that $\varphi \equiv 0$ and $\psi(x) = \epsilon g(x)$ with $\epsilon > 0$. If $g(x) \geq 0$ for all $x \in \mathbb{R}$, and $\int_{\delta/2}^{\delta} g(y)dy > 0$ with some $\delta \in (0,1)$, then there exist constants $\epsilon_0 > 0$ and $C > 0$ such that*

$$T_* \leq C\epsilon^{-p^2} \quad \text{for } 0 < \epsilon \leq \epsilon_0. \tag{2.3}$$

It suffices to show that the solution to the following integral equation blows up in finite time:

$$u(x,t) = u_0(x,t) + \frac{1}{2} \iint_{\Gamma(x,t)} \frac{|u(y,s)|^p}{(1+y^2)^{(a+1)/2}} dyds, \quad (x,t) \in \mathbb{R} \times [0,\infty). \tag{2.4}$$

In order to prove that the solution to (2.4) blows up in finite time, we prepare a couple of lemmas below. For $l \geq 1$, we set $\Sigma(l) = \{(x,t) \in [0,\infty)^2 : t - x \geq l\}$.

Lemma 2.2. *Let* $u \in C(\mathbb{R} \times [0,\infty))$ *be the solution of* (2.4). *Then there exists a constant* $C_1 > 0$*, independent of* ϵ*, such that*

$$u(x,t) \geq C_1 \epsilon^p, \quad (x,t) \in \Sigma(1). \tag{2.5}$$

Lemma 2.3. *Let* $u \in C(\mathbb{R} \times [0,\infty))$ *be the solution of* (2.4) *and let* $L > 0$*. If we set* $T_1 = \max\{3, 2 + 2\langle 1\rangle^{a+1}(C_1)^{-p}\epsilon^{-p^2}L\}$*, then we have* $u(x,t) \geq L$ *for* $(x,t) \in \Sigma(T_1)$*.*

Lemma 2.4. *Let* $u \in C(\mathbb{R} \times [0,\infty))$ *be the solution of* (2.4). *Let* A*,* $T > 0$ *and* $0 < h \leq 1$*. If we set* $A' = 2^{-1}\langle 1\rangle^{-(a+1)}A^p h^2$ *and* $T' = T + 2h$*, then*

$$u(x,t) \geq A, \quad (x,t) \in \Sigma(T) \tag{2.6}$$

implies

$$u(x,t) \geq A', \quad (x,t) \in \Sigma(T'). \tag{2.7}$$

3 Nonlinear Klein-Gordon equation in de Sitter spacetime

In this section, we are interested in the initial value problem for the semilinear Klein-Gordon equation in the de Sitter spacetime,

$$\begin{aligned} &\partial_t^2\Phi + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = F(\Phi), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ &\Phi(x,0) = \varphi_0(x), \quad \partial_t\Phi(x,0) = \varphi_1(x), \qquad x \in \mathbb{R}^n, \end{aligned} \tag{3.1}$$

where $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$, F is a smooth function and $m > 0$ is called physical mass.

In the Minkowski spacetime, the initial value problem for the semilinear Klein-Gordon equation

$$u_{tt} - \Delta u + m^2 u = |u|^\alpha u,$$

has been extensively investigated. The existence of global weak solutions has been obtained by Jörgens [7], Pecher [13], Brenner [4], Ginibre and Velo [5, 6]. In order to use that the total energy remains constant for this equation, one needs the assumption $\alpha < 4/(n-1)$. On the other hand, the initial value problem for so-called Higgs boson equation

$$u_{tt} - \Delta u - m^2 u = -|u|^\alpha u, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}$$

in the Minkowski spacetime, and

$$\partial_t^2\Phi + nH\Phi_t - e^{-2Ht}\Delta\Phi - m^2\Phi = -|\Phi|^\alpha \Phi, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}$$

in the de Sitter spacetime are studied by Yagdjian [16], and some qualitative property of the solution revealed if the global solution exists. In addition, it was shown in Baskin [2] that the initial value problem for

$$\partial_t^2 u + n\partial_t u + \frac{\partial_t \sqrt{h_t}}{\sqrt{h_t}}\partial_t u + e^{-2t}\Delta_{h_t} u + \lambda u + |u|^\alpha u = 0, \quad (y,t) \in Y \times \mathbb{R}$$

admits a small amplitude global solution in the energy space $H^1 \oplus L^2$, provided $\lambda > n^2/4$ and $\alpha = 4/(n-1)$. Here h is a smooth family of Riemannian metrices on compact n-dimensional manifold Y, which is characterized as an asymptotically de Sitter spacetime. The cases $(\alpha, n) = (4,3)$ and $(2,4)$ are also considered by Baskin [1].

Turning back to the initial value problem (3.1), Yagdjian [17] showed the global existence in the Sobolev space $H^s(\mathbb{R}^n)$ provided $m \in (0, \sqrt{n^2-1}/2) \cup [n/2, \infty)$, $\|\varphi_0\|_{H^s(\mathbb{R}^n)}\| + \|\varphi_1\|_{H^s(\mathbb{R}^n)}$ is sufficiently small, $s > n/2$ and $F(\Phi) = |\Phi|^\alpha \Phi$ where α is a positive even integer or $\alpha > s$.

In Nakamura [12], the assumption on the regularity of the initial data is weakened in the case of large mass, i.e., $m \geq n/2$. On the contrary, we are interested in the case of small mass, that is, $0 < m < \sqrt{n^2-1}/2$, and wish to strengthen the decay property of the global solution. We give another proof based on the L^∞ estimate.

Theorem 3.1. *Let $0 < m < \sqrt{n^2-1}/2$, k be positive integer satisfying $[([n/2] + 1 + k)/2] + 1 \leq k$ and $F(\Phi) = |\Phi|^\alpha \Phi$ where α is a positive even integer. Assume that φ_0, $\varphi_1 \in W^{N,2}(\mathbb{R}^n)$ with $N := [n/2] + 1 + k$ and their supports supp$\varphi_0 \subset \mathbb{R}^n$, supp$\varphi_1 \subset \mathbb{R}^n$ are compact. Then, there are constants $\epsilon_0 > 0$ and $R > 0$ such that if $\|\varphi_0\|_{W^{N,2}(\mathbb{R}^n)} + \|\varphi_1\|_{W^{N,2}(\mathbb{R}^n)} \leq \epsilon$ for $0 < \epsilon \leq \epsilon_0$, the problem in* (3.1) *has a solution $\Phi \in C([0,\infty); W^{k,\infty}(\mathbb{R}^n))$ satisfying $e^{\left(\frac{n}{2}-M\right)t}\|\Phi(.,t)\|_{W^{k,\infty}(\mathbb{R}^n)} \leq R\epsilon$ where $M := \sqrt{\frac{n^2}{4} - m^2}$.*

In order to prove the theorem, we need the L^∞ estimates for the solutions of the linear Klein-Gordon equation with and without source term.

We introduce the fundamental solutions for the linear Klein-Gordon equation in the de Sitter spacetime and give a representation of its solution. For $(x_0, t_0) \in \mathbb{R}^{n+1}$ we define the forward and backward light cones as follows:

$$D_\pm(x_0,t_0) := \left\{(x,t) \in \mathbb{R}^{n+1} : \ |x - x_0| \leq \pm(e^{-t_0} - e^{-t})\right\}.$$

We define

$$\begin{aligned} E(x,t;x_0,t_0;M) := {} & (4e^{-t_0-t})^{-M} \left((e^{-t} + e^{-t_0})^2 - (x-x_0)^2\right)^{-\frac{1}{2}+M} \\ & \times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-t_0} - e^{-t})^2 - (x-x_0)^2}{(e^{-t_0} + e^{-t})^2 - (x-x_0)^2}\right), \end{aligned}$$

for $(x,t) \in D_+(x_0,t_0) \cup D_-(x_0,t_0)$, where $M = \sqrt{\frac{n^2}{4} - m^2}$. Here we used the notation $(x-x_0)^2 = (x-x_0).(x-x_0)$ for $x, x_0 \in \mathbb{R}^n$. The kernels $K_0(z,t;M)$ and $K_1(z,t;M)$ are defined by

$$K_0(z,t;M) := -\left[\frac{\partial}{\partial b}E(z,t;0,b;M)\right]_{b=0},$$

and $K_1(z,t;M) := E(z,t;0,0;M)$, that is

$$K_1(z,t;M) := (4e^{-t})^{-M}\left((1+e^{-t})^2 - z^2\right)^{-\frac{1}{2}+M} F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\right).$$

The kernels $K_0(z,t;M)$ and $K_1(z,t;M)$ play an important role in the derivation of basic estimates for the linear Klein-Gordon equation in the de Sitter spacetime.

It was shown in [15] that the solution $\Phi = \Phi(x,t)$ of the initial value problem

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = 0, \quad \Phi(x,0) = \varphi_0(x), \quad \Phi_t(x,0) = \varphi_1(x), \tag{3.2}$$

with φ_0, $\varphi_1 \in C_0^\infty(\mathbb{R}^n)$ is given by

$$\begin{aligned}\Phi(x,t) &= e^{-\frac{n-1}{2}t}v_{\varphi_0}(x,\phi(t))\\ &\quad + e^{-\frac{n}{2}t}\int_0^1 v_{\varphi_0}(x,\phi(t)s)\left(2K_0(\phi(t)s,t;M) + nK_1(\phi(t)s,t;M)\right)\phi(t)ds\\ &\quad + e^{-\frac{n}{2}t}\int_0^1 v_{\varphi_1}(x,\phi(t)s)(2K_1(\phi(t)s,t;M))\phi(t)ds,\end{aligned} \tag{3.3}$$

where $\phi(t) := 1 - e^{-t}$ with $t > 0$. Here, for $\varphi \in C_0^\infty(\mathbb{R}^n)$, $v_\varphi(x,t)$ denotes the solution of

$$v_{tt} - \Delta v = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty). \tag{3.4}$$

Moreover, the solution $\Phi = \Phi(x,t)$ of

$$\Phi_{tt}+n\Phi_t-e^{-2t}\Delta\Phi+m^2\Phi = f, \quad \Phi(x,0) = 0, \quad \Phi_t(x,0) = 0, \quad (x,t) \in \mathbb{R}^n\times(0,\infty), \tag{3.5}$$

with $f \in C^\infty(\mathbb{R}^{n+1})$ is given by

$$\Phi(x,t) = 2e^{-\frac{n}{2}t}\int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{n}{2}b}v(x,r;b)E(r,t;0,b;M), \tag{3.6}$$

where $v(x,t;b)$ is the solution to the following initial value problem for the wave equation

$$v_{tt} - \Delta v = 0, \quad v(x,0;b) = f(x,b), \quad v_t(x,0;b) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{3.7}$$

where $b > 0$.

We derive L^∞ estimates for the linear Klein-Gordon equation in the de Sitter spacetime.

Theorem 3.2. *Let $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ and $0 < m < \frac{\sqrt{n^2-1}}{2}$. Then the solution $\Phi = \Phi(x,t)$ of* (3.2) *satisfies the following estimate*

$$\|\Phi(.,t)\|_{L^\infty(\mathbb{R}^n)} \leq Ce^{(M-\frac{n}{2})t}\left(\|\varphi_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} + \|\varphi_1\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}\right), \tag{3.8}$$

for all $t \in (0,\infty)$. Here we put $M = \sqrt{\frac{n^2}{4} - m^2}$.

Theorem 3.3. *Let $f \in C^\infty(\mathbb{R}^{n+1})$ and $0 < m < \frac{\sqrt{n^2-1}}{2}$. Then the solution $\Phi = \Phi(x,t)$ of* (3.5) *satisfies the following estimate*

$$\|\Phi(.,t)\|_{L^\infty(\mathbb{R}^n)} \leq Ce^{-(\frac{n}{2}-M)t}\int_0^t e^{(\frac{n}{2}-M)b}\|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}\,db, \tag{3.9}$$

for all $t \in (0,\infty)$. Here we put $M = \sqrt{\frac{n^2}{4} - m^2}$.

Since the local smooth solution of (3.1) exists, we need to derive a suitable apriori estimate for proving the global solvability of (3.1). Let k be a positive integer satisfying $[([n/2]+1+k)/2]+1 \leq k$. We assume that the solution of (3.1) satisfies

$$e^{\left(\frac{n}{2}-M\right)t}\|\Phi(.,t)\|_{W^{k,\infty}(\mathbb{R}^n)} \leq R\epsilon \quad \text{for} \quad t \in [0,T), \tag{3.10}$$

where $R, T > 0$ and $\epsilon > 0$.

In order to estimate the nonlinear term $F(\Phi)$ we use the following lemma.

Lemma 3.4. *Let $F(\Phi) = |\Phi|^\alpha \Phi$ with an even integer $\alpha > 0$ and let k be as above. Then we have*

$$\|F(\Phi)\|_{W^{N,1}(\mathbb{R}^n)} \leq C\|\Phi\|^\alpha_{W^{k,\infty}(\mathbb{R}^n)}\|\Phi\|_{W^{N,2}(\mathbb{R}^n)}, \tag{3.11}$$

where C is a positive constant.

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