ON THE FOURIER COEFFICIENTS OF HILBERT MODULAR FORMS OF HALF INTEGRAL WEIGHT OVER ARBITRARY ALGEBRAIC NUMBER FIELDS

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We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. We write F for an algebraic number field, $\mathfrak{d}$ for the different of F relative to $\mathbb{Q}$, $\mathfrak{o}$ for the integral ring of F. F has r_1 real archimedian primes and r_2 imaginary archimedian primes. $\sigma_i : F \to \mathbb{R}$ $(1 \le i \le r_1)$ are the mutually distinct embeddings of F to $\mathbb{R}$, and $\sigma_{r_1+j} : F \to \mathbb{C}$ $(1 \le j \le r_2)$ are the mutually distinct imaginary conjugate embeddings of F to $\mathbb{C}$ such that $\sigma_{r_1+j} \ne \overline{\sigma_{r_1+l}}$, $\sigma_{r_1+j} \ne \sigma_{r_1+l}$ $(1 \le j, l \le r_2,\ j \ne l)$, $\sigma_{r_1+j} \ne \overline{\sigma_{r_1+j}}$ $(1 \le j \le r_2)$ and $\sigma_i \ne \sigma_{r_1+j}$ $(1 \le i \le r_1, 1 \le j \le r_2)$. For $\alpha \in F$, we put

$$\alpha^{(i)} = \sigma_i(\alpha) \text{ and } \alpha^{(r_1+j)} = \sigma_{r_1+j}(\alpha) \quad (1 \le i \le r_1, 1 \le j \le r_2).$$

Let $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the Hamilton quaternion algebra, $\tilde{H} = \{\mathfrak{z} = z + wj \in \mathbb{H} | z \in \mathbb{C}, w > 0\}$, $H = \{z = x + iy | x \in \mathbb{R}, y > 0\}$ and $D = H^{r_1} \times \tilde{H}^{r_2}$. $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ (resp., $\mathrm{GL}_2^+(\mathbb{R})$) acts on $\tilde{H}$ (resp., H) in the following way

$$\mathfrak{z} \mapsto g(\mathfrak{z}) = (a'\mathfrak{z} + b')(c'\mathfrak{z} + d')^{-1} \in \tilde{H} \quad \left(\mathfrak{z} \in \tilde{H} \text{ and } \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \frac{1}{\sqrt{\det g}} g\right)$$

$$\left(\text{resp.,} \quad z \mapsto g(z) = (az + b)(cz + d)^{-1} \in H \quad (z \in H)\right).$$

$(\mathrm{GL}_2(\mathbb{R}))^{r_1} \times (\mathrm{GL}_2(\mathbb{C}))^{r_2}$ acts on D in the normal way.

$$\mu_0(g, \mathfrak{z}) = c\mathfrak{z} + d \text{ and } m(g, \mathfrak{z}) = |\mu_0(g, \mathfrak{z})|^2 \quad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \text{ and } \mathfrak{z} \in \tilde{H}\right).$$

We define the Laplace-Beltrami operator $L_{\mathfrak{z}}$ on $\tilde{H}$ by

$$L_{\mathfrak{z}} = w^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial w^2} \right) - w \frac{\partial}{\partial w} \quad (\mathfrak{z} = z + jw \text{ and } z = x + iy).$$

We define the differential operator $L_z^{m+\frac{1}{2}}$ on H by $L_z^{m+\frac{1}{2}}(f) = 4\delta^{m+\frac{1}{2}-2}\varepsilon f$, where $\varepsilon f = -y^2 \frac{\partial f}{\partial \bar{z}}$ and $\delta^{m-\frac{3}{2}} f = y^{-m+\frac{3}{2}} \frac{\partial (y^{m-\frac{3}{2}} f)}{\partial z}$.

$$G = \mathrm{SL}_2(F) \text{ and } \tilde{G} = \mathrm{GL}_2(F).$$

Let $\mathfrak{x}$ and $\mathfrak{y}$ be the factorial ideal such that

$$D[\mathfrak{x},\mathfrak{y}] = G_A \cap (\tilde{G}_{a^+} \prod_{v\in h} \mathfrak{o}[\mathfrak{x},\mathfrak{y}]_v^\times), \qquad \tilde{D}[\mathfrak{x},\mathfrak{y}] = \tilde{G}_{a^+} \prod_{v\in h} \mathfrak{o}[\mathfrak{x},\mathfrak{y}]_v^\times$$
$$\text{and } \mathfrak{o}[\mathfrak{x},\mathfrak{y}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,d\in\mathfrak{o}, b\in\mathfrak{x}, c\in\mathfrak{y} \right\}.$$

The theta function $\vartheta(\mathfrak{z})$ attached to F is given by

$$\vartheta(\mathfrak{z}) = \sum_{b\in\mathfrak{o}} e_s(b^2z/2)e_{\mathbb{C}}(b^2z/2)\exp(-2\pi w|b|^2) \; (\mathfrak{z} = (z_1,\dots,z_{r_1},\mathfrak{z}_{r_1},\dots,\mathfrak{z}_{r_1+r_2}) \in D),$$

where $\mathfrak{z}_{r_1+i} = z_{r_1+i} + jw_{r_1+i}$, $e_s(b^2z/2) = \prod_{i=1}^{r_1} e[(b^{(i)})^2z_i/2]$, $e_{\mathbb{C}}(b^2z/2) = \prod_{i=1}^{r_2} e[-\Re((b^{(r_1+i)})^2z_{r_1+i})]$, $\exp(-2\pi w|b|^2) = \prod_{i=1}^{r_2}\exp(-2\pi|b^{(r_1+i)}|^2w_{r_1+i})$, and $e[z] = \exp(2\pi iz)$. Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two integral ideals of F.

$$\Gamma = \Gamma[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{b}'\mathfrak{d}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \middle| a,d\in\mathfrak{o}, b\in 2\mathfrak{b}\mathfrak{d}^{-1} \text{ and } c\in 2\mathfrak{b}'\mathfrak{d} \right\}.$$

Then ϑ satisfies $\vartheta(\gamma(\mathfrak{z})) = h(\gamma,\mathfrak{z})\vartheta(\mathfrak{z})$ ($\forall\gamma\in\Gamma[2\mathfrak{d}^{-1},2\mathfrak{d}]$ and $\forall\mathfrak{z}\in D$), where $h(\gamma,\mathfrak{z})$ is the automorphic factor given by Shimura. For a $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma[2\mathfrak{d}^{-1},2\mathfrak{d}]$, we put $J_m(\gamma,\mathfrak{z}) = h(\gamma,\mathfrak{z})^{-3}\prod_{i=1}^{r_1}(c_\gamma^{(i)}z_i + d_\gamma^{(i)})^{m_i+2}\prod_{i=1}^{r_2} m(\gamma^{(r_1+i)},\mathfrak{z}_{r_1+i})^3$, $m = (m_1,\dots,m_{r_1}) \in \mathbb{Z}^{r_1}$, $m_i > 1$ $(1\le i\le r_1)$ and $u_{r_1} = (1,\dots,1)\in\mathbb{Z}^{r_1}$. Let Ψ be the Hecke Größen-character of F and $\mathfrak{f}$ be the conductor with $\mathfrak{f}|4\mathfrak{b}\mathfrak{b}'$. Define $\mathscr{S}_{m+\frac{1}{2}u_{r_1},\omega}(\mathfrak{b},\mathfrak{b}';\Psi) = \Big\{ f : D\to\mathbb{C}(C^2\text{-function}) \Big|$ (i) $f(\gamma(\mathfrak{z})) = \Psi_{\mathfrak{f}}(a_\gamma)J_m(\gamma,\mathfrak{z})f(\mathfrak{z})$ $(\forall\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma[2\mathfrak{b}\mathfrak{d}^{-1},2\mathfrak{b}'\mathfrak{d}]$, $\forall\mathfrak{z}\in D)$; (ii) $L_{z_i}^{m_i+\frac{1}{2}}f(\mathfrak{z}) = \omega_i f(\mathfrak{z})$ $(1\le i\le r_1)$, $L_{\mathfrak{z}_{r_1+i}}(w_{r_1+i}^{\frac{3}{2}}f(\mathfrak{z})) = \omega_{r_1+i}w_{r_1+i}^{\frac{3}{2}}f(\mathfrak{z})$ $(1\le i\le r_2)$; (iii) f vanishes at every cusp of $\Gamma[2\mathfrak{b}\mathfrak{d}^{-1},2\mathfrak{b}'\mathfrak{d}]\Big\}$. In this case, the Fourier expansion of f is as follows: For a $\beta\in G\cap\mathrm{diag}[\gamma,\gamma^{-1}]D[2\mathfrak{b}\mathfrak{d}^{-1},2\mathfrak{b}'\mathfrak{d}]$ with $\gamma\in F_{\mathbb{A}}$, then

$$\begin{aligned}
&\Psi_a(d_\beta)\Psi^*(d_\beta A_\beta^{-1})f(\beta^{-1}(\mathfrak{z}))N(A_\beta)^{\frac{1}{2}} \\
=&J_m(\beta,\beta^{-1}(\mathfrak{z}))^{-1}\sum_{\xi\in F^\times}\mu_f(\xi,A_\beta^{-1})\prod_{i=1}^{r_1} e[\xi^{(i)}\Re(z_i)]W_{\alpha_i,\beta_i}(\xi^{(i)}\Im(z_i)) \\
&\times\prod_{i=1}^{r_2} e[\Re(\xi^{(r_1+i)})z_{r_1+i}]\prod_{i=1}^{r_2}\left(4\pi\left|\frac{\xi^{(r_1+i)}}{2}w_{r_1+i}\right|^{-\frac{1}{2}}\right)K_{\nu_i}(2\pi|\xi^{(r_1+i)}w_{r_1+i}|),
\end{aligned}$$

where $\mathfrak{z} = (z_1,\dots,z_{r_1},\mathfrak{z}_{r_1+1},\dots,\mathfrak{z}_{r_1+r_2})$, $\mathfrak{z}_{r_1+i} = z_{r_1+i}+jw_{r_1+i}$, $\alpha_i+\beta_i = 1-(m_i+\frac{1}{2})$, $\alpha_i\beta_i = \omega_i$ $(1\le i\le r_1)$, W_{α_i,β_i} is the Whittaker function, $\nu_i^2 = 1+\omega_{r_1+i}$ $(1\le i\le r_2)$, K_{ν_i} is the Bessel function and $A_\beta = \gamma^{-1}\mathfrak{o}$.

For every prime ideal $\mathfrak{p}$ in F, we can define Hecke operator $\mathbb{T}_{\mathfrak{p}}$ on the space above. We can also define the space $\mathcal{S}_{n,\omega}(\mathfrak{i},\Psi)$ of modular forms with even weight, and define the Hecke operator $\mathfrak{T}_{\mathfrak{i}}$ on this space for every integral ideal $\mathfrak{i}$.

Let $f \in \mathscr{S}_{m+\frac{1}{2}u_{r_1},\omega}(\mathfrak{b},\mathfrak{b}',\Psi)$, $m=(m_1,\dots,m_{r_1})$ $(m_i>1\ (1\le i\le r_1))$, $\tau\in F^\times$ with $\tau \gg 0$, $\tau\mathfrak{b}=\mathfrak{q}^2\mathfrak{r}$ (fractional ideal $\mathfrak{q}$, and square free integral ideal $\mathfrak{r}$). We put $\mathfrak{c}=4\mathfrak{b}\mathfrak{b}'$ and $\varphi=\Psi\varepsilon_\tau$, where ε_τ is Hecke character associated to the extension $F(\sqrt{\tau})/F$. We put $\tilde{G}_{\mathbb{A}}=\bigsqcup_{\lambda=1}^{\kappa}\tilde{G}\,\mathrm{diag}\,[1,t_\lambda]\tilde{D}[\mathfrak{d}^{-1},2\mathfrak{b}\mathfrak{b}'\mathfrak{d}]$ $(t_\lambda\in F_h^\times)$. We choose $h\in\mathscr{S}_{m+\frac{1}{2}u_{r_1},\omega}(\mathfrak{o},\mathfrak{r}\mathfrak{b}\mathfrak{b}';\varphi)$ satisfying $\mu_h(\xi,\mathfrak{m})=\mu_f(\tau\xi,(\mathfrak{q}\mathfrak{r})^{-1}\mathfrak{m})$.

Define a function $g_{\tau,\lambda}(\mathfrak{w})=\Psi_{\tau,\lambda}(f)(\mathfrak{w})$ $(1\le\lambda\le\kappa)$ on D by

$$Cg_{\tau,\lambda}(\mathfrak{w})=\int_{\Gamma_{\mathfrak{r}\mathfrak{c}}\backslash D} h(\mathfrak{z})\Theta(\mathfrak{z},\mathfrak{w},\eta_\lambda)(\Im z)^{m+\frac{1}{2}u_{r_1}}w^3 d\mathfrak{z},$$

where $d\mathfrak{z}=\prod_{i=1}^{r_1}\frac{dx_i dy_i}{y_i^2}\prod_{i=1}^{r_2}\frac{dx_{r_1+i}dy_{r_1+i}dw_{r_1+i}}{w_{r_1+i}^3}$ with $\mathfrak{z}\in D$, C is a constant number, $\Gamma_{\mathfrak{r}\mathfrak{c}}=\Gamma[2\mathfrak{d}^{-1},2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}]$ and $\Theta(\mathfrak{z},\mathfrak{w},\eta_\lambda)$ is some theta function. We deduce the following theorem.

Theorem 1. *$\{\Psi_{\tau,\lambda}(f)(\mathfrak{w})\}_{\lambda=1}^{\kappa}\in\mathcal{S}_{2m,\omega'}(2\mathfrak{b}\mathfrak{b}',\Psi^2)$ $(\omega'=(4\omega_1,\dots,4\omega_{r_1},4\omega_{r_1+1}+3,\dots,4\omega_{r_1+r_2}+3))$. The Fourier expansion of $\Psi_{\tau,\lambda}(f)(\mathfrak{w})$ is*

$$\Psi_{\tau,\lambda}(f)(\mathfrak{w})=N\Big(\frac{t_\lambda}{\mathfrak{r}}\Big)\sum_{\mathfrak{m}}\sum_{\substack{l\in t_\lambda\mathfrak{r}^{-1}\mathfrak{m}\\ (lt_\lambda^{-1}\mathfrak{m}^{-1}\mathfrak{r},\mathfrak{r}\mathfrak{c})=1}} N(\mathfrak{m})l^{m-1}|l|^{-1}\varphi_a(l)\varphi^*(l\mathfrak{r}/t_\lambda\mathfrak{m})\mu_f(\tau,(\mathfrak{r}\mathfrak{q})^{-1}\mathfrak{m})$$

$$\times e_s(l\Re(z))e_c(lu)\prod_{i=1}^{r_1}c(\mathrm{sgn}(l^{(i)}))W_{2\alpha,2\beta}(l\Im(z))vK_{2\nu}(4\pi|l|v),$$

where $\mathfrak{m}$ runs over all integral ideals, $\mathfrak{w}=(z_1,\dots,z_{r_1},\mathfrak{z}_{r_1+1},\dots,\mathfrak{z}_{r_1+r_2})$, $z=(z_1,\dots,z_{r_1})$, $\mathfrak{z}_{r_1+i}=u_{r_1+i}+jv_{r_1+i}$ $(1\le i\le r_2)$, $u=(u_{r_1+1},\dots,u_{r_1+r_2})$, $v=(v_{r_1+1},\dots,v_{r_1+r_2})$, $l^{m-1}=\prod_{i=1}^{r_1}(l^{(i)})^{m_i-1}$, $|l|=\prod_{i=1}^{r_2}|l^{(r_1+i)}|$, φ_a is the restriction of φ on the archimedian part of the adelization of $F^\times$, φ^* is the ideal character such that $\varphi^*(t\mathfrak{o})=\varphi(t)$ if $t\in F_v^\times$ and $\varphi(\mathfrak{o}_v^\times)=1$, $e_s(l\Re(z))=\prod_{i=1}^{r_1}e[l^{(i)}\Re(z_i)]$, $e_c(lu)=\prod_{i=1}^{r_2}e[2\Re(l^{(r_1+i)}u_{r_1+i})]$, $W_{2\alpha,2\beta}(l\Im(z))=\prod_{i=1}^{r_1}W_{2\alpha_i,2\beta_i}(l^{(i)}\Im(z^{(i)}))$, $vK_{2\nu}(4\pi|l|v)=\prod_{i=1}^{r_2}v_{r_1+i}K_{2\nu_i}(4\pi|l^{(r_1+i)}|v_{r_1+i})$, $c(\mathrm{sgn}(y))=1$ if $y>0$ and $c(\mathrm{sgn}(y))=0$ otherwise.

We deduce the following theorem.

Theorem 2. *Suppose that f is a Hecke eigen form satisfying multiplicity one theorem. Then*

$$|\mu_f(\tau,\mathfrak{q}^{-1})|^2\mu(\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c})=R\varphi^*(\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c})^{-1}N(\mathfrak{r}\mathfrak{q})^{-1}D(0,\chi,\bar{\varphi})\langle f,f\rangle/\langle\mathfrak{g},\mathfrak{g}\rangle,$$

where $\mathfrak{g}$ *is the primitive form associated with* $\{\Psi_{\tau,\lambda}(f)\}_{\lambda=1}^{\kappa}$*,* χ *are Hecke eigenvalues of* $\mathfrak{g}$*,* $D(s,\chi,\bar{\varphi})$ *is L-series associated with* χ *and quadratic character* $\bar{\varphi}$*, and*

$$\begin{aligned} R =&\pi^{-\{m\}}2^{-1+(r_1/2)+2r_2-\{m\}}\tau_s^{m+(1/2)u_{r_1}}\pi^{-r_1/2}|\tau_c|^3 \\ &\times \Gamma'(\alpha+m)\Gamma'(\beta+m)\Gamma'(\nu+1/2)\Gamma'(-\nu+1/2)[\mathfrak{o}_+^\times : (\mathfrak{o}^\times)^2]h_F \\ &\times \frac{\mathrm{vol}(\Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}]\backslash D)}{\mathrm{vol}(\Gamma[2\mathfrak{r}\mathfrak{d}^{-1}, \mathfrak{c}\mathfrak{d}]\backslash D)} \sum_{\mathfrak{t}\supset\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}} \mu(\mathfrak{t})\varphi^*(\mathfrak{t})\chi(\mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}). \end{aligned}$$

Shimura proved the theorem above in the case where f is holomorphic modular form of half integral weight over totally real algebraic number field, and he proposed to extend his theorem to the case where f is modular form of half integral weight over arbitrary algebraic number field.

We use the following formulas.

$$\int_0^\infty v^{\frac{\varepsilon-3}{2}}\exp(-\alpha v-\beta v^{-1})H_\varepsilon(\sqrt{2\alpha v})dv = \beta^{\frac{\varepsilon-1}{2}}\sqrt{\pi}2^{\frac{\varepsilon}{2}}\exp(-2\sqrt{\alpha\beta}) \quad (\alpha,\beta>0),$$

$$\int_0^\infty \exp(\frac{p}{t})\exp(-at)K_s(at)t^{-\frac{3}{2}}dt = 2\sqrt{\pi}p^{-\frac{1}{2}}K_{2s}(2\sqrt{2}\sqrt{ap}) \quad (a,p>0)$$

and
$$\begin{aligned} &\int_0^\infty y^l K_{s'}(y)K_{s''}(y)dy \\ =&2^{l-2}\Gamma((l+s'+s''+1)/2)\Gamma((l-s'+s''+1)/2)\Gamma((l+s'-s''+1)/2) \\ &\Gamma((l-s'-s''+1)/2)\;/\;\Gamma(l+1). \end{aligned}$$