Singularities of several geometric objects related to null Cartan curve in Minkowski 3-space

Donghe Pei and Zhigang Wang

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P.R.China

1. Introduction

If we imagine a regular curve in $\mathbb{R}_1^3$, denote γ, and then imagine the set of all principal normal lines intersecting this curve. Unless γ is a line, these lines will all meet some locus, in fact it is the locus of centres of curvatures of γ, which we call the *focal sets*. It is obvious that the focal set would be a point or a new curve depending on how our original given curves. Focal sets are useful to the study of certain optical phenomena (namely scattering, in fact a rainbow is caused by caustics), expressing some geometrical results within fluid mechanics as well as describing many medical anomalies [1–4], and so it is important to study the geometric properties related to the *focal curve*(i.e., the locus of focal set is a curve) of a curve.

It is well known that there exist spacelike curve, timelike curve and null curve in Minkowski spacetime. For non-null curve in Minkowski space, many of the classical results from Riemannian geometry have Lorentz counterparts. In fact, spacelike curves or timelike curves can be studied by approaches similar to those taken in positive definite Riemannian geometry. Non-null curves in Minkowski space, regarding singularity, have been studied extensively by, among others, the second author and by Izumiya et al.[5–10]. The importance of the study of null curves and its presence in the physical theories are clear [4, 11–18]. Nersessian and Ramos [19] also show us that there exists a geometrical particle model based entirely on the geometry of the null curves in Minkowskian 4-dimensional spacetime which under quantization yields the wave equations corresponding to massive spinning particles of arbitrary spin. They have also studied the simplest geometrical particle model which is associated with null curves in Minkowski 3-space[20]. However, null curves have many properties which are very different from spacelike and timelike curves[11, 21, 22]. In other words, null curve theory has many results which have no Riemannian analogues. In geometry of null curves difficulties arise because the arc length vanishes, so that it is impossible to normalize the tangent vector in the usual way. Bonner introduces the Cartan frame as the most useful one and he uses this frame to study the behaviors of a null curve[23]. Thus, one can use these fundamental results as the basic tools in researching the geometry of null curves. However, to the best of the authors' knowledge, the singularities of surfaces and curves as they relate to null Cartan curves(see Section 2) have not been considered in the literature, aside from our studies in de Sitter 3-space [24, 25]. Thus, the current study hopes to serve such a need, in this article, we study *the focal surfaces* and *the binormal indicatrix* associated with a null Cartan curve in Minkowski 3-space from the standpoint of singularity theory .

2. Preliminaries

Let $\mathbb{R}_1^3$ denote the 3-dimensional Minkowski space, that is to say, the manifold $\mathbb{R}^3$ with a flat Lorentz metric $\langle,\rangle$ of signature $(-,+,+)$, for any vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, we set $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3$. We

also define a vector

$$x \wedge y = \begin{vmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the canonical basis of $\mathbb{R}^3_1$. We say that a vector $x \in \mathbb{R}^3_1 \backslash \{\mathbf{0}\}$ is *spacelike, null* or *timelike* if $\langle x, x \rangle$ is positive, zero or negative, respectively. The *norm* of a vector $x \in \mathbb{R}^3_1$ is defined by $\| x \| = \sqrt{|\langle x, x \rangle|}$. We call x a *unit vector* if $\| x \| = 1$.

Let $\gamma : I \to \mathbb{R}^3_1$ be a smooth regular curve in $\mathbb{R}^3_1$ (i.e., $\dot{\gamma}(t) \neq 0$ for any $t \in I$), parametrized by an open interval I. For any $t \in I$, the curve γ is called a *spacelike curve*, a *null(lightlike) curve* or a *timelike curve* if all its velocity vector satisfy $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0, \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$ or $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle < 0$, respectively. We call γ a *non-null curve* if γ is a timelike curve or a spacelike curve.

Let $\gamma : I \to \mathbb{R}^3_1$ be a null curve in $\mathbb{R}^3_1$ (i.e., $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$ for any $t \in I$). Now suppose that γ is framed by a null frame . A null frame $F = \{\xi = \frac{d\gamma}{dt}, N, B\}$ at a point of $\mathbb{R}^3_1$ is a positively oriented 3-tuple of vectors satisfying

$$\langle \xi, \xi \rangle = \langle B, B \rangle = 0, \quad \langle \xi, B \rangle = 1,$$

$$\langle \xi, N \rangle = \langle B, N \rangle = 0, \quad \langle N, N \rangle = 1.$$

The Frenet formula of γ with respect to F is given by

$$\begin{cases} \frac{d\xi}{dt} & = -h\xi + k_1 N \\ \frac{dB}{dt} & = hB + k_2 N \\ \frac{dN}{dt} & = -k_2 \xi - k_1 B. \end{cases} \tag{2.1}$$

The functions h, k_1 and k_2 are called the *curvature functions* of γ (cf.[11]). Employing the usual terminology, the spacelike unit vector filed N will be called the *principal normal vector filed*. The null vector filed B is called the *binormal vector filed*. Null frames of null curves are not uniquely determined. Therefore, the curve and a frame must be given together.

There always exists a parameter s of γ such that $h = 0$ in Eqs.(2.1). This parameter is called a *distinguished parameter* of γ, which is uniquely determined for prescribed screen vector bundle (i.e. a complement in $\langle \frac{d\gamma}{dt} \rangle^{\perp}$ to $\langle \frac{d\gamma}{dt} \rangle$) up to affine transformation[11].

Let $\gamma(s)$ be a null curve with a distinguished parameter in $\mathbb{R}^3_1$(i.e. $h = 0$ in Eqs.(2.1)). Moreover we assume that $\gamma'(s), \gamma''(s), \gamma'''(s)$ are linearly independent for all s. Then we consider the basis $E = \{\gamma'(s), \gamma''(s), \gamma'''(s)\}$ such that $\langle \gamma''(s), \gamma''(s) \rangle = k_1(s) = 1$. We choose the $\xi = \frac{d\gamma}{ds}, N = \gamma''$, then there exists only one null frame $F = \{\xi, N, B\}$ for which $\gamma(s)$ is a framed null curve with Frenet equations [11]:

$$\begin{cases} \frac{d\xi}{ds} = N \\ \frac{dB}{ds} = k_2 N \\ \frac{dN}{ds} = -k_2 \xi - B, \end{cases} \tag{2.2}$$

where $\xi = \frac{d\gamma}{ds}, N = \gamma'', B = -\gamma''' - k_2\gamma', k_2 = \frac{1}{2}\langle \gamma''', \gamma'' \rangle$. We call Eqs.(2.2) the *Cartan Frenet equations* and $\gamma(s)$ their *null Cartan curve*[11]. We remark that the curvature function k_2 is an invariant under Lorentzian transformations.

In case γ is a null Cartan curve, labeling $k_2(s) = k(s)$, then the Frenet formula of $\gamma(s)$ with respect to $F = \{\xi, N, B\}$ becomes

$$\begin{cases} \frac{d\xi}{ds} = N(s) \\ \frac{dB}{ds} = k(s)N(s) \\ \frac{dN}{ds} = -k(s)\xi(s) - B(s). \end{cases} \tag{2.3}$$

This frame satisfies

$$\xi(s) \wedge B(s) = N(s), \quad N(s) \wedge \xi(s) = \xi(s), \quad B(s) \wedge N(s) = B(s).$$

Now we define surface $\mathcal{FS} : I \times \mathbb{R} \to \mathbb{R}_1^3$ by

$$\mathcal{FS}(s, \mu) = \gamma(s) + \frac{1}{k(s)}N(s) + \mu B(s).$$

We call $\mathcal{FS}(s, \mu)$ the *Focal surface* of null Cartan curve γ. We define the 2-dimensional *future lightcone* vertex at v_0 by

$$LC_+^*(v_0) = \{x \in \mathbb{R}_1^3 : \langle x - v_0, x - v_0 \rangle = 0, x_0 > 0\}.$$

When v_0 is the null vector $\mathbf{0}$, we simply denote $LC_+^*(\mathbf{0})$ by LC_+^*.

Let $\gamma : I \to \mathbb{R}_1^3$ be a regular null Cartan curve. We define the *binormal normal indicatrix* of $\gamma(s)$ as the map $\mathcal{BI}_\gamma : I \to LC_+^*$ given by

$$\mathcal{BI}_\gamma(s) = B(s)$$

and *the focal curve* of $\gamma(s)$ as the map $\mathcal{F}_\gamma : I \to \mathbb{R}_1^3$ given by

$$\mathcal{F}_\gamma(s) = \gamma(s) + \frac{1}{k(s)}N(s).$$

Defining the set: for any $v_0 \in \mathbb{R}_1^3$, $TPB(v_0) = \{u \in \mathbb{R}_1^3 | \langle u - v_0, B(s) \rangle = 0\} \backslash \{v_0\}$, we call it the *tangential planar bundle of lightcone through* v_0. It is obvious that the lightcone $LC_+^*(v_0)$ is the envelope of the tangential planar bundle.

We give a geometric invariant σ of a null Cartan curve in $\mathbb{R}_1^3$ by

$$\sigma(s) = k^3(s) + 3k'^2(s) - k(s)k''(s),$$

which are related to the geometric meanings of the singularities of the focal surface.

We shall assume throughout the whole article that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

3. Volumelike distance function and lightcone height function of null Cartan curve

The purpose of this section is to obtain one geometric invariants of null Cartan curves by constructing a family of functions of the null Cartan curve.

Let $\gamma : I \to \mathbb{R}_1^3$ be a regular null Cartan curve with $k(s) \neq 0$. We define a three-parameter family of smooth functions

$$D : I \times \mathbb{R}_1^3 \to \mathbb{R}$$

by $D(s, v) = |B(s)\ N(s)\ \gamma(s) - v| = \langle \gamma(s) - v, B(s) \rangle$. Here, $|a\ b\ c|$ denotes the determinant of matrix $(a\ b\ c)$. We call D the *volumelike distance function* of null Cartan curve γ. We denote that $d_v(s) = D(s, v)$ for any fixed vector v in $\mathbb{R}_1^3$. Using Eqs.(2.3) and making a simple calculation, we can state the following facts.

Proposition 3.1. *Suppose* $\gamma : I \to \mathbb{R}_1^3$ *is a regular null Cartan curve with* $k(s) \neq 0$ *and* $v \in \mathbb{R}_1^3$. *Then*
(1) $d_v(s) = 0$ *if and only if there exist real numbers* λ, ω *such that* $\gamma(s) - v = \mu B(s) + \omega N(s)$.
(2) $d_v(s) = d_v'(s) = 0$ *if and only if* $v = \gamma(s) + (1/k(s))N(s) - \mu B(s)$.
(3) $d_v(s) = d_v'(s) = d_v''(s) = 0$ *if and only if* $v = \gamma(s) + (1/k(s))N(s) + (k'(s)/k^3(s))B(s)$.
(4) $d_v(s) = d_v'(s) = d_v''(s) = d_v'''(s) = 0$ *if and only if* $v = \gamma(s) + (1/k(s))N(s) + (k'(s)/k^3(s))B(s)$ *and* $\sigma(s) = k^3(s) + 3k'^2(s) - k(s)k'(s) = 0$.
(5) $d_v(s) = d_v'(s) = d_v''(s) = d_v'''(s) = d_v^{(4)} = 0$ *if and only if* $v = \gamma(s) + (1/k(s))N(s) + (k'(s)/k^3(s))B(s)$ *and* $\sigma(s) = \sigma'(s) = 3k^2(s)k'(s) + 5k(s)k''(s) - k^2(s)k'''(s) = 0$.

Let $\gamma : I \to \mathbb{R}_1^3$ be a regular null Cartan curve. We define a two-parameter family of functions

$$H : I \times LC_+^* \to \mathbb{R}$$

by $H(s, v) = \langle \gamma(s), v \rangle - s$. We call H the *lightcone height functions* of null Cartan curve $\gamma(s)$. We denote that $h_v(s) = H(s, v)$ for any fixed vector v in LC_+^*. We have the following proposition.

Proposition 3.2. *Suppose $\gamma : I \to \mathbb{R}^3_1$ is a regular null Cartan curve and $v \in LC^*_+$. Then*
(1) $h'_v(s) = 0$ if and only if there exist real numbers λ, ω such that $v = \lambda\xi(s) + B(s) + \omega N(s)$ and $2\lambda + \omega^2 = 0$.
(2) $h'_v(s) = h''_v(s) = 0$ if and only if $v = B(s)$.
(3) $h'_v(s) = h''_v(s) = h'''_v(s) = 0$ if and only if $v = B(s)$ and $k(s) = 0$.
(4) $h'_v(s) = h''_v(s) = h'''_v(s) = h^{(4)}_v(s) = 0$ if and only if $v = B(s)$ and $k(s) = k'(s) = 0$.

Proposition 3.3. *Let $\gamma : I \to \mathbb{R}^3_1$ be a regular null Cartan curve with $k(s) \neq 0$. Then*
(1) The singularities of $\mathcal{FS}$ is the set $\left\{ (s, \mu) | \mu = \frac{k'(s)}{k^3(s)}, s \in I \right\}$.
(2) If $\mathcal{FS}(s, \frac{k'(s)}{k^3(s)}) = v_0$ is a constant vector, then $\mathcal{F}_\gamma(s)$ is in $TPB(v_0)$ for any s in I and $\sigma(s) = k^3(s) + 3k'^2(s) - k(s)k''(s) \equiv 0$.

This work is only a preparation for further studying, in the following, we will discuss some geometrical properties of null Cartan curve from singularity theory viewpoint.

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