Progress of Theoretical Physics, Vol. 111, No. 4, April 2004

Two-Dimensional Ising Models with Layered Quenched Bond Randomness II.

—— *Boundary Magnetization and Its Universal Critical Behavior* ——

Kazuyuki MATSUMOTO[1,*)] and Yasuhiro AKUTSU[2,**)]

[1] *Department of Applied Sciences, Muroran Institute of Technology, Muroran 050-8585, Japan*

[2] *Department of Physics, Graduate School of Science, Osaka University, Toyonaka 560-0043, Japan*

(Received)

This paper investigates the generalized McCoy-Wu type random Ising model. It is found that boundary magnetization behaves as $(T_c - T)^{\beta'}$, $\beta' = 1.0$ just below the 'critical temperature'. From numerical analyses it is revealed that this type of critical behavior, $\beta' = 1.0$, would be universal for any type of randomness. This universal behavior is explained in terms of the destruction of the devil's staircase structure. As an observable result of this model, the integrated distribution function of boundary magnetization is suggested.

§1. Introduction and historical review

In the past thirty years, much effort has been done to reach some understandings of an ordered phase in disordered magnetic materials.[1)] However, it seems that a theoretical consensus is not achieved yet. In the research of random spin systems, exact solution models are convincing. One such model is the one McCoy and Wu(MW) discovered in 1968, the two-dimensional exact solution model.[2)] In this model, the vertical bonds can vary row to row randomly, but must be the same along each row. We will refer to this model as the MW model hereafter.

In the original work of MW, they succeeded in showing that the critical singularity of the specific heat's divergence is weakened: it becomes infinitely differentiable, but non-analytic at the 'critical temperature'. Furthermore, in subsequent papers, they discussed the spin correlation function,[3)] and McCoy discussed boundary magnetization.[4)] In particular, McCoy showed that the singular behavior of this system still holds for interactions with the generalized distribution function.[5)] We can say that it is the general delta function in this case. These results are summarized in an excellent review and book.[6)] After these works, Au-Yang and McCoy found that the leading singularity for the pure system vanishes when the sample average is taken out.[7),8)]

Extensions of the MW model has been tried by several authors. In this extension, the horizontal bonds also vary as the vertical bonds, i.e. the horizontal bonds vary row to row randomly, but must be the same along each row. Hence, this model

*) E-mail: matumoto@mmm.muroran-it.ac.jp

**) E-mail: acts@phys.sci.osaka-u.ac.jp

is called the generalized MW (GMW) model in this paper. This extension is so attractive, because it may give insight into the spin glass problem. That is, the GMW model can include the frustration effect.[9)]

In 1980, Longa obtained an exact expression of the 'critical temperature' of the GMW model.[10)] In our previous work, we derived the explicit expression of free energy in a exact manner.[11)] In 1987, Shanker and Murthy studied the critical singularity of the GMW model in detail.[12)] They showed that the GMW model exhibits the Griffith singularity. Nieuwenhuizen and Orland also studied the similar model.[14)]

In spite of above the detailed studies, the nature of the GMW model as well as the original MW model have not been clarified yet. In particular, the distributions of the physical quantities are interesting and not so clear even at present. We would like to discuss the relation between the devil's staircase (DS) structure and the GMW type random spin system.

Here, we summarize the results of our previous work.[11)] It was shown that the 'critical temperature' defined by MW coincides with the peaked temperature of the specific heat. However, the peaked value of the specific heat is not divergent even in an infinite system thus producing a different result to that of McCoy.[5)] Further, it was also shown that the distribution function of the random variables, which are induced by two-dimensional random matrices, has the DS structure. We also found that thermal fluctuations destroy this structure in a region of the θ-T phase diagram. Here, θ is an integral variable and T is the temperature, and we set $k_B = 1$. This behavior should be regard as a 'phase transition' of the distribution function. Furthermore, we determined the phase boundary of this 'phase transition' in the θ-T phase diagram.

This paper is organized as follows: the GMW type random Ising model is defined in §2. A formulation to obtain free energy of this model is also developed in the same section. Boundary magnetization, and its numerical analyses are given in §3. Phase diagrams of the DS structure associated with boundary magnetization are discussed in §4. The results of the Monte Carlo simulation are also shown in §4. The integrated distribution functions of the internal energy and the specific heat are shown in §5. Finally, §6 is devoted to comments and summary.

§2. The GMW type random Ising model and its free energy

In this paper, we consider the two-dimensional random Ising model on the square lattice. This lattice has M(vertical)$\times$ N(horizontal) lattice points, and is periodic in the horizontal direction. We set the boundary condition to be free for the vertical direction. Ising spin on this lattice interacts with the coupling strength $J^V(j)$ (vertical) and $J^H(j)$ (horizontal), where j denotes the row. Thus, we can write the model Hamiltonian of the GMW model as follows,

$$H = -\sum_{i,j} J^V(j)\sigma_{i,j}\cdot\sigma_{i,j+1} - \sum_{i,j} J^H(j)\sigma_{i,j}\cdot\sigma_{i+1,j}, \tag{2·1}$$

where the spin variable at the site i,j takes the value $\sigma_{i,j} = \pm 1$. When the vertical bonds are random, we call this model V-type which is the original MW model itself. In the same manner, we refer to H-type and VH-type according to their type of randomness.

We have several methods to obtain the well known Onsager solution of the two-dimensional Ising model. Below, we derive the exact solution of the GMW model using the Pfaffian method following the original work of MW.

The partition function of the GMW model can be expressed by the Pfaffian of the antisymmetric $4MN \times 4MN$ matrix A. Further, from the mathematical identity $(PfA)^2 = detA$, the square of the partition function is expressed as

$$Z^2 = detA, \tag{2·2}$$

where

$$A(j,k:j,k) = \begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}, \tag{2·3}$$

$$A(j,k:j,k+1) = -A^T(j,k+1:j,k) = \begin{pmatrix} 0 & z_j^H & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2·4}$$

$$A(j,k:j+1,k) = -A^T(j+1,k:j,k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_j^V \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2·5}$$

In the above,

$$z_j^H = \tanh \beta J^H(j), \tag{2·6}$$

and

$$z_j^V = \tanh \beta J^V(j), \tag{2·7}$$

where $\beta = 1/T$.

Using the translational symmetry of the system, i.e. z_j^H and z_j^V depend only on suffix j (not on i), the right hand side of Eq.(2·2) can be transformed into the product of determinants

$$Z^2 = \prod_{\theta} detB(\theta), \tag{2·8}$$

where the $4M \times 4M$ matrices $B(\theta)$ are given by

$$B(j,j:\theta) = \begin{pmatrix} 0 & 1+z_j^H e^{i\theta} & -1 & -1 \\ -1-z_j^V e^{-i\theta} & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}, \tag{2·9}$$

and

$$B(j,j+1:\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_j^V \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2·10}$$

with $\theta = \pi(2n-1)/N \ \ (n = 1, 2, \cdots, N)$. We can reduce the matrix $B(\theta)$ to the product of 2×2 random matrices for each θ. Thus, we can evaluate Eq.(2·8) using random variable $y_j(\theta)$ which is defined in the following recursion relation,

$$y_{j+1}(\theta) = \frac{a_j(\theta) + z_j^{V^2} y_j(\theta)}{a_j^2(\theta) + b_j^2(\theta) + a_j(\theta) z_j^{V^2} y_j(\theta)}, \tag{2·11}$$

where

$$a_j(\theta) = -2z_j^H sin\theta \mid 1 + z_j^H e^{i\theta} \mid^{-2}, \tag{2·12a}$$

$$b_j(\theta) = (1 - z_j^{H^2}) \mid 1 + z_j^H e^{i\theta} \mid^{-2}, \tag{2·12b}$$

with $y_0(\theta) = 0$.

Finally, free energy of the GMW model is obtained as

$$-\beta F = \lim_{M\to\infty} \frac{1}{M}[\sum_{j=1}^{M} \ln\{2\cosh(\beta J^V(j))\} + \sum_{j=1}^{M} \ln\{2\cosh(\beta J^H(j))\}]$$
$$+ \lim_{M\to\infty} \frac{1}{4\pi M}[\sum_{j=1}^{M}\int_0^{2\pi} d\theta \ln\{1 + z_j^{H^2} + 2z_j^H \cos\theta\}$$
$$+ \sum_{j=1}^{M}\int_0^{2\pi} d\theta \ln\{a_j^2(\theta) + b_j^2(\theta) + a_j(\theta) z_j^{V^2} y_j(\theta)\}]. \qquad (2\cdot 13)$$

§3. Boundary magnetization

3.1. *Derivation of boundary magnetization formula*

In the preceding section, exact free energy of the GMW model is obtained, but thermodynamic quantities which are derived from this free energy are limited to the internal energy and the specific heat. By the use of the same method which is developed in §2, we can also obtain boundary magnetization. Boundary magnetization may become a directly observable quantity in future experiments. Hence, its distribution will be important in the research of random magnetic systems. Thus, we think that the study of boundary magnetization is not so pedantic.

To obtain boundary magnetization, a magnetic field h is applied to the M-th row. The partition function of such a system ($M\times N$ lattice with boundary magnetic field h) is equivalent to one of a new lattice which includes an additional row which is connected to the original lattice by bonds of strength z, where $z = \tanh(\beta h)$. For convenience we define the variable $c = -2\sin\theta \mid 1 + e^{i\theta} \mid^{-2}$. Therefore, to obtain boundary magnetization the problem is reduced to calculating the exact expression of free energy for the new $(M+1)\times N$ lattice. Such free energy is obtained in the same manner in §2. This bond arrangement is shown in Fig. 1.[13]

In this way, the value of boundary magnetization is given by the derivative with

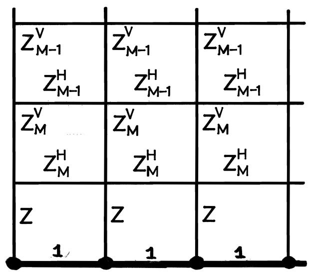


Fig. 1. The McCoy-Wu type random Ising lattice: the boundary field h is applied to the last row, and we use the bond strength $z = tanh\beta h$. The parameter z is a coupling constant for the vertical bond.

respect to the applied magnetic field h,

$$M_b = -\frac{\partial F(h)}{\partial h} = z + \frac{1}{2\pi}\int_0^{2\pi} s(y_M) d\theta, \tag{3·1}$$

where

$$s(x) = \frac{z(1-z^2)x}{c + z^2 x}. \tag{3·2}$$

Here, in this integral the near $\theta = 0$ has an important contribution. Then, we can expand $s(y_M)$ near $\theta = 0$, and extend the integral limits to infinity. We also take the limit $z \to 0$. Thus, we obtain the compact formula of spontaneous boundary magnetization,

$$M_b = \frac{1}{2}\sqrt{\frac{\prod_{j=1}^{M} z_j^{V^2} w_j^+ - \prod_{j=1}^{M} w_j^-}{\sum_{i=1}^{M} \prod_{j=1}^{i} z_j^{V^2} \prod_{j=1}^{i-1} w_j^+ z_i^H \prod_{j=i+1}^{M} w_j^-}}, \tag{3·3}$$

where

$$w_j^+ = (1 + z_j^H)^2, \tag{3·4a}$$

$$w_j^- = (1 - z_j^H)^2. \tag{3·4b}$$

3.2. *Numerical analysis for the critical exponent β'*

In the following, we restrict ourselves to the ferromagnetic random spin system to clarify the issue. We can determine the 'critical temperature' T_c at which the boundary magnetization becomes non-zero. This is given by

$$\prod_j z_{jc}^{V^2} = \prod_j \frac{w_{jc}^-}{w_{jc}^+}, \tag{3·5}$$

where z_{jc}^V, w_{jc}^- and w_{jc}^+ are the critical values at the 'critical temperature'. Boundary magnetization just below the 'critical temperature' is described as

$$M_b \sim (T_c - T)^{\beta'}, \tag{3·6}$$

where β' is the critical exponent for boundary magnetization. For the pure system, we can derive from Eq.(3·3) that boundary magnetization behaves as $(T_c - T)^{1/2}$, i.e. $\beta' = 1/2$. On the other hand, for the GMW model, $\beta' = 1.0$ is concluded as follows.

(1) It is noted that we must take the sample average over many sample realizations. Here, we consider the V-type GMW model for simplicity. Random bonds are generated according to the distribution function,

$$P(J^V(j)) = (1-p)\delta(J^V(j) - J_1^V) + p\delta(J^V(j) - J_2^V), \tag{3·7}$$

where p is the concentration of $J_2^V (> J_1^V > 0)$. From the numerical analysis, we found that boundary magnetization is well plotted along a straight line just below the 'critical temperature', i.e. $\beta' = 1.0$, see Fig. 2.

(2) For the ferromagnetic Gaussian randomness, which distribution function is given by

$$P(J^V(j)) = (\sqrt{\frac{\pi\Delta}{2}})^{-1} \cdot exp(-\frac{J^V(j)^2}{2\Delta}), \tag{3·8}$$

with $J^V(j) > 0$, we meet $\beta' = 1.0$. This value is consistent with the McCoy's result, $\beta' = 1$.[4)]

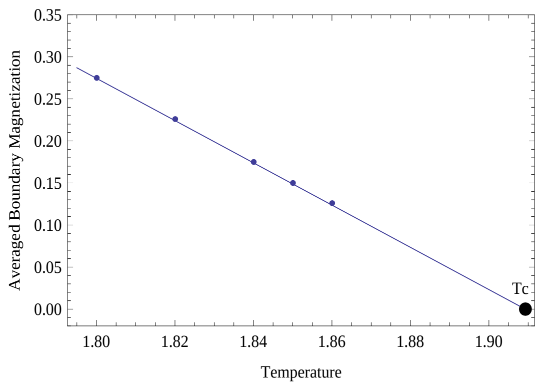


Fig. 2. Boundary magnetization versus temperature: This figure is obtained by Eq.3·1. From this figure, we can realize $\beta' = 1.0$. Numerical error for $\beta' = 1.0$ seems rather small. The system size is 512×512. Bond distribution is binary type: $\frac{1}{2}(\delta(J^V(j) - 0.5) + \delta(J^V(j) - 1.0))$. The exact T_c is 1.90924. We take an average with 20-samples. The boundary magnetization is set $h = 0.005$.

Hence, it seems that $\beta' = 1.0$ is universal for any type of ferromagnetic bond randomness. In the next section, we will give evidence of this universality, that is $\beta' = 1.0$ may be independent of the bond distribution function.

§4. The devil's staircase structure

As shown in the preceding section, we can calculate boundary magnetization for ferromagnetic binary type bond randomness using Eq.(3·1), (3·2) and Eq.(3·7). By the sample average, we can infer $\beta' = 1.0$ for binary type of randomness. Furthermore, the same conclusion holds for ferromagnetic Gaussian randomness Eq.(3·8). In this section, this universal nature is explained in terms of the DS structure for $s(y_M)$.

4.1. *The devil's staircase structure for $s(y_M)$*

As is emphasized in the previous report, a noticeable property of the MW type random Ising model is the DS structure of the integrated distribution function $\nu(y)$.[11] In addition to $\nu(y)$, we will discuss $s(y_M)$ itself. We can develop the argument of Bruntuma and Aeppli to an integrated distribution function of $s(y_M)$.[15),16] We define the integrated distribution function $N(s)$ by

$$N(s) = \int_{-\infty}^{s} ds' << \delta(s' - s(y_M)) >>, \tag{4·1}$$

where $<< \cdots >>$ denotes the sample average. As in §3.2, we restrict ourselves to V-type bond randomness and take the binary type distribution function Eq.(3·7) for simplicity. Hence, we can show that this function $N(s)$ satisfies the following recursion relation,

$$N(s) = (1 - p)N(g_1(s)) + pN(g_2(s)), \tag{4·2}$$

where

$$g_1(x) = s^{-1}(x; J_1^V), \tag{4·3a}$$

$$g_2(x) = s^{-1}(x; J_2^V). \tag{4·3b}$$

In the above, $s^{-1}(x; J_1^V)$ and $s^{-1}(x; J_2^V)$ are the inverse functions of $s(x)$ with J_1^V and J_2^V, respectively. It can be shown that this recursion relation leads to the DS structure for $N(s)$. As in our previous paper this structure is destroyed by thermal fluctuations and becomes a smooth function in the region of θ-T phase diagram, however.[11] We can determine the phase boundary which separates the DS phase and non-DS phase by

$$g_1(\tfrac{1}{2}) = g_2(\tfrac{1}{2}). \tag{4·4}$$

These results are shown in Figs. 3(a) and (b) for typical binary bond randomness with some h.

We will illuminate the reason for the universal critical behavior, i.e. the universal critical exponent $\beta' = 1.0$ is concluded for any type of randomness. The DS structure

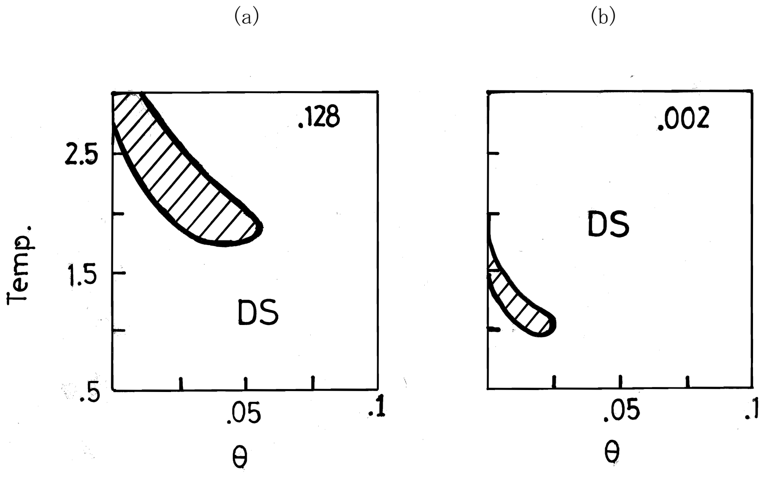


Fig. 3. Phase diagram of θ-T: The phase boundary which separates the DS phase and non-DS phase is shown. Boundary fields h are (a) 0.128 and (b) 0.002, respectively. Bond randomness is the same in Fig. 2.

of $N(s)$ is originated from the binary bond distribution function. It destroys and may become a universal function in the limit of $h \to 0$ and $\theta \to 0$. For example, if we take the ferromagnetic Gaussian bond distribution function, we may get a integrated distribution function $N(s)$. Next, if we take the binary bond distribution function, we may reach the *same* function $N(s)$ in that limit. Thus, even if the bond distribution function is any type, in the non-DS region $N(s)$ may become the universal function in that limit. As is shown in Fig. 3(b) the shaded region (the non-DS phase) would collapse to the $\theta = 0$ line in the limit of $h \to 0$. Furthermore, it should be stressed that the criticality of spontaneous boundary magnetization comes from the contribution of near $\theta = 0$ and $h = 0$. This is the same in a pure system.

From these two observations, we may be led to the following conclusion. *The dominant contribution of the integral of boundary magnetization comes from one of the destroyed distribution functions.* This would induce the universal critical behavior, $\beta' = 1.0$.

4.2. *Sample realization of the devil's staircase-like structure*

Though the existence of the DS structure of $s(y_M)$ is fascinating, this DS structure cannot be accessible experimentally. So, an experimentally observable quantity is desirable. In one such quantity, a distribution function of boundary magnetization with sample realization would satisfy this requirement.

Using Eq.(3·1) we can obtain boundary magnetization for binay type bond randomness Eq.(3·7). In this calculation, the integrated distribution function of boundary magnetization can be obtained for the sample realization. The typical result is shown in Fig. 4. From this figure it is realized that the integrated distribution function of boundary magnetization has a step. This means that in the distribution function of boundary magnetization itself there are two peaks with a gap.

In the real magnetic materials, samples have a 3D structure. We therefore can conceive that when these two-dimensional GMW models are stacked vertically, the lateral sides become surfaces of the sample. If the surface magnetization has a similar step (or two peaks) experimentally, one can identify the distribution of the interactions as ferromagnetic binary type bond randomness. It should be noted that, even in this randomness, the critical exponent is universal, i.e. $\beta' = 1.0$. We hope both a signature of binary type randomness and $\beta' = 1.0$ are observed at the same time experimentally in the future.

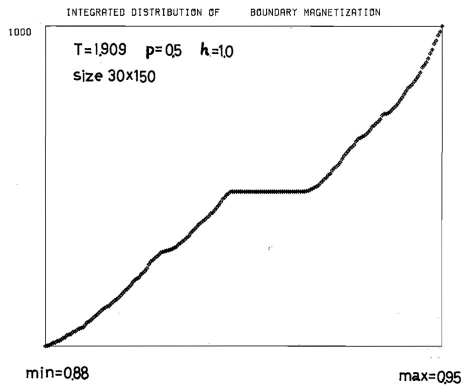


Fig. 4. The integrated distribution function with respect to the sample realization for boundary magnetization. The system size is 30×150. Bond randomness is the same in Fig. 2. Calculation is done at $T = 1.909$ and $h = 1.0$. The number of samples is 1000.

4.3. *Monte Carlo Simulation*

To obtain bulk magnetization, we carried out a Monte Carlo simulation for the model in §3.2, i.e. $p = 0.5$, $J_1^V = 0.5$. The system size is 64×64, the number of samples is 467 and the simulation was done at $T = 1.5$. In this simulation, the site magnetizations are averaged within 26-35 layers. In Fig. 5, we can see some steps clearly. See Fig. 5 and its caption for the detail.

Next, layered magnetizations are also calculated. The first layered $m_M = M_b$ (boundary magnetization, $M = 63$), the second layered m_{M-1} and the third layered m_{M-2} are shown as their integrated distribution functions in Figs. 6(a)-(c). From these Figs. 6(a)-(c), we can realize that the integrated distribution functions of layered magnetizations also have steps in which the smaller index of m, the narrower the steps become.

§5. Integrated Distributions of the Internal Energy and the Specific Heat

The integrated distributions of the internal energy which is deduced from free energy presented in Eq.(2·13) are shown in Figs. 7(a)-(c) for each temperature.

The stepwise structure is observed at a lower temperature $T = 1.0$. However, at the 'critical temperature' $T = 1.909$, this integrated distribution structure disappears. When the temperature is raised further, the integrated distribution structure

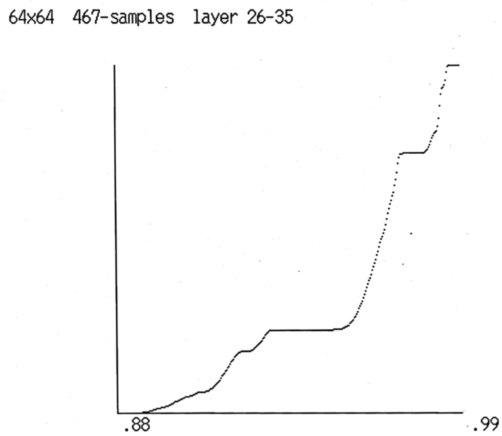
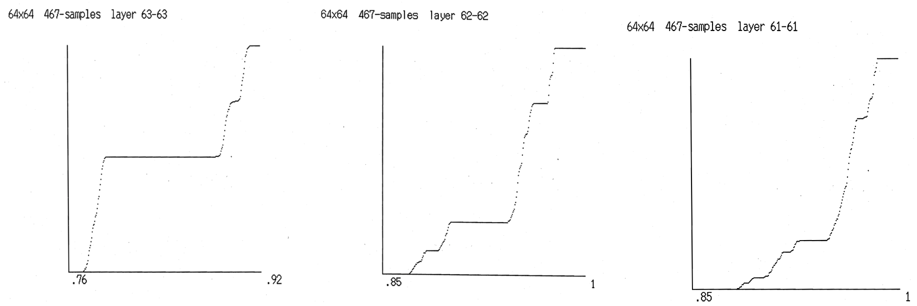


Fig. 5. The integrated distribution function for bulk magnetization using a Monte Carlo simulation. Bond randomness is the same in Fig. 2. Calculation is done at $T = 1.5$ and $m_{M+1} = 1$, i.e. $(M+1)$ layered spins are fixed, where $M = 63$. For pure systems, bulk magnetizations are given exactly. These are $m_{min} = 0.8849$ and $m_{max} = 0.9865$ for $p = 0.0$ and $p = 1.0$, respectively. So, the averaged magnetizations for each sample distribute between them.



(a) (b) (c)

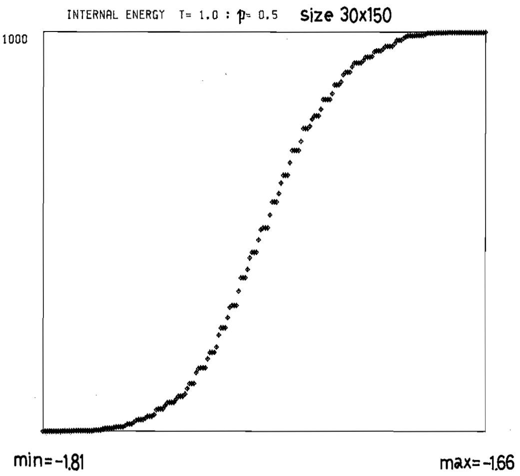
Fig. 6. (a) The integrated distribution functions for the first layered (boundary) magnetization, (b) for the second layered magnetization, (c) for the third layered magnetization. Bond randomness is the same in Fig. 2. Conditions of this simulation are the same in Fig. 5.

is recovered, see Fig. 7(c). This reentrant phenomena is remarkable.

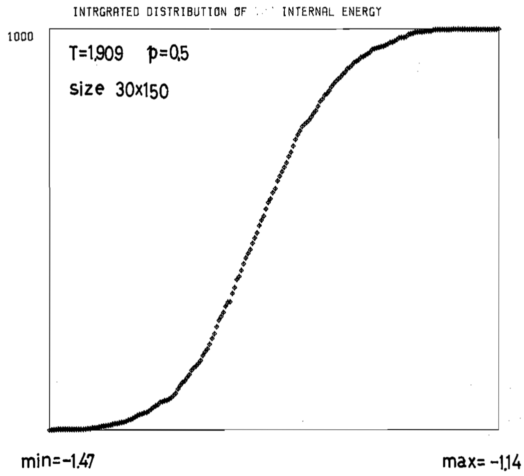
The specific heat is also deduced from Eq.(2·13). The numerical results for several temperatures are shown in Figs. 8(a)-(d). For lower temperatures the integrated distribution functions are smooth, whereas at higher temperature the integrated distribution function's structure is fine. This phenomena may be observed in experiments which measure the specific heat precisely.

§6. Comments and Summary

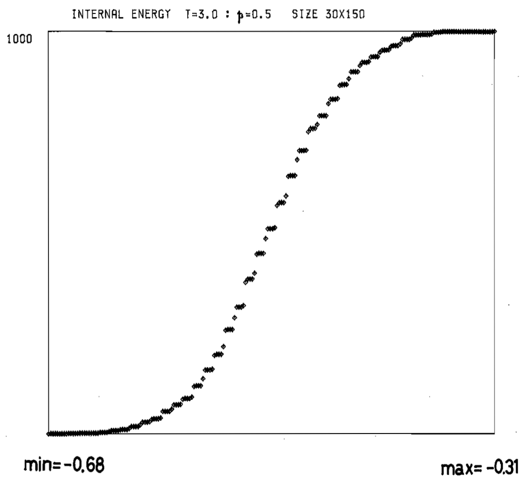
We would like to make three comments:



(a)



(b)



(c)

Fig. 7. (a) The integrated distribution function for the internal energy at $T = 1.0$, (b) $T = 1.909$ and (c) $T = 3.0$. Bond randomness is the same in Fig. 2. The system size is 30×150. The number of samples is 1000.

(1)We have examined the periodic frustrated models (non random) which were investigated by Hoever, Wolff and Zittartz (HWZ).[17] The GMW model, in specific instances, becomes the periodic H-type model. We will comment further on the below.

From Eq.(3·3), it is obvious that the critical exponent of boundary magnetization for all such models (HWZ models) is the same for the pure lattice, i.e. $\beta' = 1/2$. This conclusion had already been noted by Ohno and Okabe.[18]

(2)In that literature, another important phenomena is observed, i.e. the reen-

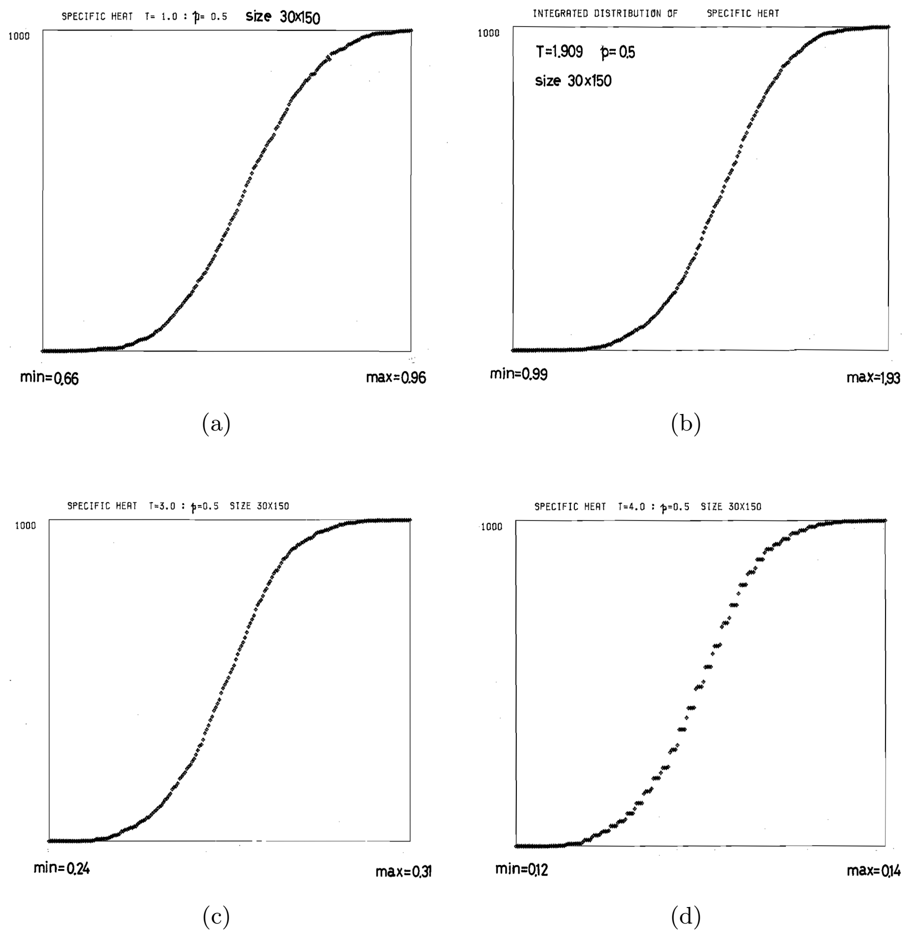


Fig. 8. (a) The integrated distribution functions for the specific heat at $T = 1.0$, (b) $T = 1.909$, (c) $T = 3.0$ and (d) $T = 4.0$. Bond randomness is the same in Fig. 2. The system size is 30×150. The number of samples is 1000.

trant behavior of boundary magnetization for the fully frustrated model. With the use of Eq.(3·3), we can calculate boundary magnetization for various HWZ models. From the results of these examples, the following conclusion would hold: *only the fully frustrated model* exhibits the reentrant behavior.

(3)In our previous paper, we showed that the specific heat of the GMW model does not diverge even at the 'critical temperature'. For the pure systems, $p = 0.0$ ($p = 1.0$), the specific heat diverge as $log|T - T_c|$. Hence, we have to comment about the criticality with respect to p. From $log(p)$ vs. C (peaked value) plot analysis, we can infer the divergent behavior is like $log(p)$ ($log(1 - p)$) with $p \to 0.0$ ($p \to 1.0$).

We summarize the results obtained in this paper. Exact boundary magnetization of the GMW model is obtained following MW's method with an extension. From the numerical analyses, universal critical behavior of boundary magnetization is found, i.e. $\beta' = 1.0$. It was clarified that this behavior is due to the existence of randomness, i.e. originated from the random average. We can observe that the DS structure of $s(y_M)$ is destroyed near $\theta = 0$ in the limit $h \to 0$. Hence, this universal behavior is explained qualitatively. We also observed the DS-like (stepwise) structure of the integrated distribution functions of boundary magnetization, layered magnetization, the internal energy and the specific heat.

Acknowledgements

The authors express their sincere thank to Associate Prof. Y. Iba for valuable comments and illuminating discussion. One of the authors (K. M.) expresses his thanks to Profs. S. Murayama, H. Takano and Associate Prof. N. Momono for their continual encouragement. He also thanks to Assistant Prof. E. T. Hagley for his critical reading of the manuscript.

References

1) For a review, K. Binder and A. P. Young, Rev. Mod. Phys. **58** (1987), 801.
2) B. M. McCoy and T. T. Wu, Phys. Rev. **176** (1968), 631.
3) B. M. McCoy and T. T. Wu, Phys. Rev. **188** (1969), 982.
4) B. M. McCoy, Phys. Rev. **188** (1969), 1014.
5) B. M. McCoy, Phys. Rev. **B2** (1970), 795.
6) B. M. McCoy, in Phase Transitions and Critical Phenomena, Vol.2, ed. C. Domb and M. S. Green (Academic Press, New York, 1972, 161), see also B. M. McCoy and T. T. Wu, The Two-Dimensional Ising Model (Harvard University Press, 1973, Chap. 14 and 15).
7) H. Au-Yang and B. M. McCoy, Phys. Rev. **B10** (1974), 886.
8) H. Au-Yang and B. M. McCoy, Phys. Rev. **B10** (1974), 3885.
9) G. Toulouse, Commun. Phys. **2** (1977), 115.
10) L. Longa, Physica **103A** (1980), 633.
11) Y. Akutsu and K. Matsumoto, Prog. Theor. Phys. **75** (1986), 1304.
12) R. Shankar and G. Murthy, Phys. Rev. **B35** (1987), 3671; ibid. **B36** (1987), 536.
13) This arrangement is different from ref. 6).
14) T. M. Nieuwenhuizen and H. Orland, Phys. Rev. **B40** (1989), 5094.
15) R. Bruinsma and G. Aeppli, Phys. Rev. Lett. **50** (1983) 1494.
16) D. Andelman, Phys. Rev. **B34** (1986) 6214.
17) P. Hoever, W. F. Wolff and J. Zittartz, Z. Phys. **B41** (1981) 43.
18) K. Ohno and Y. Okabe, J. Phys. Soc. Jpn. **55** (1986) 2627.